High-Frequency Measurement of the Non-Gaussian Macroeconomic Dynamics

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Abstract

This paper uses a non-Gaussian state-space model to extract daily business conditions from data observed at mixed frequencies. The model extends the framework of the ADS (Aruoba, Diebold and Scotti) index by allowing for non-Gaussian disturbances for observations. The extracted business conditions index behaves similarly to the ADS index, but can better capture some extreme economic movements. The distribution of non-Gaussian shocks is approximated by a mixture of normals. The likelihood function is computed by particle filter methods. The computation uses the latest development in graphical processing units (GPU).

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Contents

1 Introduction

Measuring macroeconomic conditions on a daily basis is an important task for central banks, financial institutions, and any other entity whose outcome depends on the state of macroeconomy. Unfortunately, not all economic data are sampled at the daily frequency. Financial data are available on a (intra-)daily basis, whereas most macroeconomic data are sampled weekly, monthly, quarterly or even annually. The unavailability of macroeconomic variables at higher frequency often leads economic analysis to be conducted at low frequency. However, policy makers, making decisions in real time, always need accurate and timely estimates of the state of economic activity. Therefore, in order to help policy makers assess the continuously evolving state of the macroeconomy, there is increasing demand for estimating the high-frequency economic dynamics using data sampled at mixed frequency.

Ghysels et al. (2006) propose using mixed data sampling (MIDAS) method, which aggregates high-frequency data to low frequency using a polynomial weighting function. As Bai et al. (2009) uncover the connection between the MIDAS regression and the Kalman Filter, Aruoba et al. (2009), propose a Gaussian state-space model to extract high-frequency business conditions from term spread, initial claims for unemployment insurance, employees on non-farm payrolls and real GDP growth data. These variables are observable daily, weekly, monthly and quarterly respectively, but the setting of the model can be easily extended to arbitrary number of variables observed at any frequency.

This paper is in a similar spirit, but uses a non-Gaussian state-space model to extract high-frequency business conditions from a variety of stock and flow data observed at mixed frequencies. It provides a more generic framework for extracting business conditions from data observed at mixed frequencies. The computation uses the latest development in graphical processing units (GPU). The extracted business condition index behaves similarly to the

ADS index, but can better capture the extremal movements of the economy. In particular, the model permits non-Gaussian disturbances for the observation equation and aims to better capture the non-Gaussian shocks to the economy[1]. The likelihood function is computed by particle filter methods. The distribution of innovations is approximated by a mixture of normal distributions.

The reminder of this article is organized as follows. Section 2 briefly reviews the modeling framework of Aruoba et al. (2009) and proposes the non-Gaussian extension. Section 3 casts the model in the state-space form and describes the particle filter algorithm. Section 4 compares the daily business condition indicator extracted from non-Gaussian state-space model with that extracted from Gaussian state-space model. Section 5 concludes.

2 The Modeling Framework

This section briefly reviews the dynamic factor model used to extract the high-frequency business condition index proposed by Aruoba et al. (2009). The model is cast at daily frequency, but can be easily extended to data at higher (intra-daily) frequency. For macroeconomic series at weekly, monthly and quarterly frequency, the daily data are treated as missing values. ADS explicitly defines the measurement equations of the stock and the flow variables using their different underlying structures of temporal aggregation. The model also allows for lagged state variables and a time trend in the measurement equations. I extend the model in two directions: first, the innovations to the measurement equations are allowed to follow a Student's-t distribution. I use a mixture of k normally distributed components to approximate the fat-tailed distribution as well as to facilitate the filter iteration. Second, the generic form of the model brings computation challenge to the model.

[1] The new version that takes into account the non-Gaussian shocks to the transition equation is forthcoming

The Dynamic Factor Model at Daily Frequency: The state of the economy is assumed to evolve at daily frequency. Much higher (intra-daily) frequencies could be used if desired. Let x_t denote underlying business conditions on day t, which evolve daily with $AR(p)$ dynamics,

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \ldots + \rho_p x_{t-p} + e_t, \tag{1}$$

where e_t is a white noise innovation with unit variance. x_t is the single factor that tracks and forecasts the real economic activity (see Stock and Watson (1989)). Let y_t^i denote the i-th daily economic or financial variable at day t. y_t^i depends linearly on x_t and possibly also on various exogenous variables and/or lags of y_t^i:

$$\begin{aligned} y_t^i = c_i \quad &+ \quad \beta_i x_t + \delta_{i1} w_t^1 + \cdots + \delta_{ik} w_t^k && (2) \\ &+ \quad \gamma_{i1} y_{t-D_i}^i + \cdots + \gamma_{in} y_{t-nD_i}^i + u_t^i. && (3) \end{aligned}$$

$w_t = w_t^1, \ldots, w_t^k$ are exogenous variables. u_t^i are contemporaneously and serially uncorrelated innovations. i is the index of the macro-financial variables incorporated in the model, $i = 1, \ldots, M$. Notice that the lags of the dependent variable y_t^i are in multiples (n) of D_i, where $D_i > 1$ is the number of days within the frequency of the observed y_t^i. nD_i lags of y_t^i are included because modeling persistence only at the daily frequency would be inadequate, as it would decay too quickly.

Temporal Aggregation: Stock versus Flows. Economic and financial variables, although evolving daily, are not actually observed daily. Let $\tilde{y}_t^i$ denote the same variable, y_t^i, observed at a lower frequency (call it the "tilde frequency"). The relationship between y_t^i and $\tilde{y}_t^i$ depends crucially on whether y_t^i is a stock or flow variable.

If y_t^i is a stock variable measured at a non-daily low frequency, the appropriate treatment is straightforward, because stock variables are simply point-in-time snapshots. At any time

t, either y_t^i is observed, in which case $\tilde{y}_t^i = y_t^i$, or it is not, in which case $\tilde{y}_t^i = NA$, where NA denotes missing data ("not available"). The stock variable measurement equation is:

$$\tilde{y}_t^i = \begin{cases} y_t^i = & c_i + \beta_i x_t + \delta_{i1} w_t^1 + \ldots + \delta_{ik} w_t^k + \gamma_{i1} y_{t-D_i}^i + \cdots + \gamma_{in} y_{t-nD_i}^i + u_t^i \\ & \text{if } y_t^i \text{ is observed.} \\ NA & \text{otherwise.} \end{cases} \tag{4}$$

Now consider flow variables. Flow variables observed at non-daily low frequencies are intra-period sums of the corresponding daily values,

$$\tilde{y}_t^i = \begin{cases} \sum_{j=0}^{D_i-1} y_{t-j}^i & \text{if } y_t^i \text{ is observed.} \\ NA & \text{otherwise,} \end{cases} \tag{5}$$

where D_i is the number of days per observational period (e.g. $D_i = 7$ if y_t^i is measured weekly). Combining this fact with Equation (2), we arrive at the flow variable measurement equation.

$$\tilde{y}_t^i = \begin{cases} \sum_{j=0}^{D_i-1} c_i + \beta_i \sum_{j=0}^{D_i-1} x_{t-j}^i + \delta_{i1} \sum_{j=0}^{D_i-1} w_{t-j}^1 + \ldots + \delta_{ik} \sum_{j=0}^{D_i-1} w_{t-j}^k \\ +\gamma_{i1} \sum_{j=0}^{D_i-1} y_{t-D_i-j}^i + \cdots + \gamma_{in} \sum_{j=0}^{D_i-1} y_{t-nD_i}^i + u_t^{*i} \\ & \text{if } y_t^i \text{ is observed.} \\ NA & \text{otherwise.} \end{cases} \tag{6}$$

$\tilde{y}_{t-D_i}^i = \sum_{j=0}^{D_i-1} y_{t-D_i-j}^i$, which is the observed flow variable on period ago (e.g. last week, last monthly, last quarter,...), and u_t^{*i} is the sum of the u_t^i over the tilde period.

Discussion of two subtleties is in order. First, note that in general D_i is time-varying, as for example the first quarter may have 90 or 91 days; the second quarter has 91 days; the third and the fourth quarter have 92 days. To simplify the notation, we ignored the difference in D_i. In the subsequent empirical implementation, D_i is in fact time-varying.

On the one hand, the first important contribution of this paper is to assume u_t^i follows a student's t distribution. The distribution of u_t^i is approximated by K normally distributed components.

$$g(u_t^i|\sigma_i^2, \lambda) = \sum_{k=1}^{K} p_k N(z_t|m_k, \nu_k^2). \tag{7}$$

where $g(\cdot|\sigma_i^2, \lambda)$ is a Student's t distribution with mean 0, variance $\sigma_i^2\lambda/(\lambda-2)$, and λ degree of freedom. $N(z_t|m_k, \nu_k^2)$ denotes the density function of a normal distribution with mean m_k and variance ν_k^2. The values of p_k, m_k and ν_k^2 are found on the basis $K = 7$ by minimizing the squared difference between $g(u_t^i|\sigma_i^2, \lambda)$ and the normal mixture, as in

$$\min_{p_k^*, m_k^*, \nu_k^{*2}} \left[g(u_t^i|\sigma_i^2, \lambda) - \sum_{k=1}^{K} p_k N(z_t|m_k, \nu_k^2) \right]^2. \tag{8}$$

On the other hand, the setting in Equation (6) requires using all lags of x_t as state variables. This leads to up to 92 state variables[2] if Equation (1) is written in state-space form as transition equations. In order to avoid the clumsiness of state variables, I adopt the method proposed in the appendix of Aruoba et al. (2009), which uses a time-varying vector denote the temporal aggregation of the state variables, x_t. Take the transition equation in modeling weekly observation for example,

$$\begin{aligned} C_{W,t} &= \zeta_t C_{W,t-1} + x_t \\ &= \zeta_t C_{W,t-1} + \rho_1 x_{t-1} + \rho_2 x_{t-2} + \ldots + +\rho_p x_{t-p} + \xi_t \\ \zeta_t &= \begin{cases} 0 & \text{if } t \text{ is the first day of the week} \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

[2]There are up to 92 days in a quarter. As the lowest frequency increases to annually, the above setting may lead to up to 365 state variables

This aggregation will facilitate the iteration in both the measurement equations and the transition equations.

State-space Representation To extract daily component x_t from date at mixed frequencies, Equation (2) – (5) are first integrated into the follow equation system. Measurement Equation:

$$\underbrace{\begin{bmatrix} \tilde{y}_t^1 \\ \tilde{y}_t^2 \\ \tilde{y}_t^3 \\ \tilde{y}_t^4 \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 \end{bmatrix}}_{\mathbf{Z}_t} \underbrace{\begin{bmatrix} x_t \\ C_{W,t} \\ C_{M,t} \\ C_{Q,t} \end{bmatrix}}_{\alpha_\mathbf{t}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \gamma_2 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \gamma_4 \end{bmatrix}}_{\mathbf{\Gamma_t}} \underbrace{\begin{bmatrix} \tilde{y}_{t-W}^2 \\ \tilde{y}_{t-M}^3 \\ \tilde{y}_{t-Q}^2 \end{bmatrix}}_{\omega_\mathbf{t}} + \underbrace{\begin{bmatrix} u_t^1 \\ \tilde{u}_t^{*2} \\ \tilde{u}_t^3 \\ \tilde{u}_t^{*4} \end{bmatrix}}_{\epsilon_\mathbf{t}}. \tag{9}$$

Using the temporal aggregation of state variables to reduce the the number of state variables, the transition equation is written as follows:

$$\underbrace{\begin{bmatrix} x_t \\ C_{W,t} \\ C_{M,t} \\ C_{Q,t} \end{bmatrix}}_{\alpha_\mathbf{t}} = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ \rho & \zeta_{W,t} & 0 & 0 \\ \rho & 0 & \zeta_{M,t} & 0 \\ \rho & 0 & 0 & \zeta_{Q,t} \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} x_{t-1} \\ C_{W,t-1} \\ C_{M,t-1} \\ C_{Q,t-1} \end{bmatrix}}_{\alpha_\mathbf{t-1}} + \underbrace{\begin{bmatrix} e_t \\ e_t \\ e_t \\ e_t \end{bmatrix}}_{\eta_\mathbf{t}}, \tag{10}$$

(8) and (9) together can be cast into the state-space form:

$$\mathbf{y_t} = \mathbf{Z_t}\alpha_\mathbf{t} + \mathbf{\Gamma_t}\omega_\mathbf{t} + \epsilon_\mathbf{t} \tag{11}$$

$$\alpha_\mathbf{t+1} = \mathbf{T}\alpha_\mathbf{t} + \mathbf{R}\eta_\mathbf{t} \tag{12}$$

$$\epsilon_\mathbf{t} \sim (0, \mathbf{H}_t) \tag{13}$$

$$\eta_\mathbf{t} \sim (0, \mathbf{Q}). \tag{14}$$

where $\mathbf{Q}$ is the variance of the daily shocks in the transition equation, $\mathbf{Q} = \sigma^2$. $\mathbf{H}_t$ is the variance-covariance matrix of the innovations to the measurement equation. For simplicity, shocks to the observed variables are assumed to be uncorrelated. In the state-space model, $\mathbf{H}_t$ is a diagonal matrix of four unknown parameters that denote the variances of the observed variables.

$$\mathbf{H}_t = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 \end{bmatrix}. \tag{15}$$

The fat-tailed state-space model further estimates the degree of freedom, λ, the parameter that controls the shape of the Student's t distribution. The variance of measure equations in the fat-tailed model is $\frac{\lambda}{(\lambda-2)} \cdot \mathbf{H}_t$. As the fat-tailed distribution is approximated by seven normal distributions, the variance of u_t^i uses the variance of normal component k with probability p_k (see equation 8).

In addition, $\mathbf{y_t}$ may contain missing values on any given day t, because each variable in $\mathbf{y_t}$ is observed at different frequency. Some variables may only be available by the end of the week, the month and quarter. To deal with the missing values, let W_t be a vector that indicates whether observations at different frequencies are available at day t. If only daily and weekly observations are available, $\mathbf{W_t} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$. If only daily and monthly observations are available, $\mathbf{W_t} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$. The measurement equation can thus be transformed into:

$$\mathbf{W_t} \cdot \mathbf{y_t} = \mathbf{W_t} \cdot \mathbf{Z_t} \alpha_\mathbf{t} + \mathbf{W_t} \cdot \mathbf{X_t} \beta + \mathbf{W_t} \cdot \mathbf{G_t} \mathbf{u_t}$$

This transformation replaces the missing observations ("NaN" value) on a particular day with 0. The filtered the business condition on day t is thus a weighted combination of observations available.

3 Estimation

Let $\psi = (\mathbf{Z}, \mathbf{\Gamma}, \sigma, \lambda)$ denote the parameters of the model. By the law of total probability, it follows that the density of the data $\mathbf{Y_t} = (y_1, \ldots, y_t)$ given ψ can be expressed as

$$f(\mathbf{Y_t}|\psi) = \prod_{s=1}^{t} f(\mathbf{Y_s}|\mathbf{Y}_{s-1}, \psi) = \prod_{s=1}^{t} \int f(\mathbf{Y_s}|\alpha_{\mathbf{s}}, \psi) f(\alpha_{\mathbf{s}}|\mathbf{Y}_{s-1}, \psi) d\alpha_{\mathbf{s}}.$$

On the one hand,

$$f(\mathbf{Y_s}|\alpha_{\mathbf{s}}, \psi) = St(\mathbf{Y_s}|\mathbf{Z_s}\alpha_{\mathbf{s}} + \mathbf{\Gamma_s}\omega_{\mathbf{s}}, \mathbf{H_s}, \lambda)$$

is the Student's-t density function with mean $\mathbf{Z_t}\alpha_{\mathbf{t}} + \mathbf{\Gamma_t}\omega_{\mathbf{t}}$, variance $\mathbf{H_t}$ and λ degrees of freedom. On the other hand, density $f(\alpha_{\mathbf{s}}|\mathbf{Y_{s-1}}, \psi)$ cannot be expressed in closed form. This makes the likelihood function of above model not easily available. Fortunately, it is possible to estimate the likelihood function by a simulation method, called particle filtering (see Chib et al. (2002)). For each t, the particle filter delivers a sample of draws on $\alpha_{\mathbf{t}}$ from the filtered distribution $f(\alpha_{\mathbf{t}}|\mathbf{Y_{t-1}}, \psi)$. These draws allow to estimate the one-step ahead density of $\mathbf{y_t}$

$$f(\mathbf{y_t}|\mathbf{Y_{t-1}}, \psi) = \int St(\mathbf{Y_t}|\mathbf{Z_t}\alpha_{\mathbf{t}} + \mathbf{\Gamma_t}\omega_{\mathbf{t}}, \mathbf{H_t}, \lambda) f(\alpha_{\mathbf{t}}|\mathbf{Y_{t-1}}, \psi) d\alpha_{\mathbf{t}}.$$

by simple Monte-Carlo averaging of $St(\mathbf{Y_t}|\mathbf{Z_t}\alpha_{\mathbf{t}} + \mathbf{\Gamma_t}\omega_{\mathbf{t}}, \mathbf{H_t}, \lambda)$ over the draws of $\alpha_{\mathbf{t}}$ from $f(\alpha_{\mathbf{t}}|\mathbf{Y_{t-1}}, \psi)$.

In particular, I consider the auxiliary particle filter introduced in Chib et al. (2002). This filter first creates a group of proposal values $\alpha_t^1, \ldots, \alpha_t^R$. These values are then reweighted to produce draws $\{\alpha_t^1, \ldots, \alpha_t^M\}$ that correspond to draws from the target distribution. Typically, we take R to be five or ten times larger than M. We now summarize the steps involved for the filter in period t.

Step 1. Generate M draws from the steady state distribution of $\alpha_{\mathbf{t}}$. To initialize the particle filter, we first simulate M trajectories of $\alpha_{\mathbf{0|0}}$ and $\mathbf{P_{0|0}}$. The next step is to propose candidate values of $\alpha_{\mathbf{1|0}}$ and $\mathbf{P_{1|0}}$ using each particle. Given $\{\alpha_{t|t}^1, \ldots, \alpha_{t|t}^j \ldots, \alpha_{t|t}^M\}$ from $(\alpha_{\mathbf{t}}|\mathbf{Y}_t, \psi)$, the one-day ahead business condition of each draw is simulated as

$$
\begin{aligned}
\alpha_{t+1|t}^j &= T\alpha_{t|t}^j + chol(P_{t|t}^j)z_t, \qquad z_t \sim N(0,1) \\
P_{t+1|t}^j &= TP_{t|t}^j T' + Q \\
v_j &= WY' - WZa_{t+1|t}^j - W\Gamma_t\omega_t \\
F_j &= (WZ)P_{t+1|t}^j Z'W' + WH_jW' \\
w_j &= \frac{\Gamma[(\lambda+p)/2]}{\pi^{1/2}\Gamma(\lambda/2)}(\lambda-2)^{-p/2}|F_j|^{1/2}\Big[1 + \frac{v_j' F_j^{-1} v_j}{\lambda - 2}^{-(\lambda+p)/2}\Big], \\
j &= 1, \ldots, M.
\end{aligned}
$$

The variance of measurement equation innovations in H_j, is formed by drawing from the K normal mixture components. q_k, m_k, ν_k^2 are selected to closely approximate the exact Student's t density with mean 0 variance σ_i^2 and degree of freedom λ. The variance-covariance matrix of ϵ_t (see equation (9)), H_j, is hence obtained by stacking the variance of $k-$th normal component with probability q_k.

Step 2. Sample R draws from the M draws of $\alpha_{\mathbf{t}}$. Sample R times the integers $1, 2, \ldots, M$ with probability proportional to $\{w_j\}_{j=1}^M$ (i.e. Each draw is drawn with probability $w_j / \sum_{j=1}^M w_j$). Let the sample indexes be $k_1 \ldots, k_j, \ldots, k_R$, and associate these with

$\alpha_{t|t}^{k_1}, \ldots, \alpha_{t|t}^{k_j}, \ldots \alpha_{t|t}^{k_R}$. k_j means the $k-$th draw in $1, \ldots, R$ takes the jth draw from 1 to M. Two arbitrary draws k_j and $(k+s)_j$ may take the value as a result of sampling with replacement. For each value of k_j from Step 1, simulate

$$
\begin{aligned}
\alpha_{t+1|t}^{k} &= T\alpha_{t|t}^{k_j} + chol(P_{t|t}^{k_j})u_t, \qquad u_t \sim N(0,1) \\
P_{t+1|t}^{k} &= TP_{t|t}^{k_j}T' + Q \\
v_k &= WY' - WZa_{t+1|t}^{*j} - W\Gamma_t\omega_t \\
F_k &= (WZ)P_{t+1|t}^{k}Z'W' + WH_kW' \\
w_k &= \frac{\Gamma[(\lambda+p)/2]}{\pi^{1/2}\Gamma(\lambda/2)}(\lambda-2)^{-p/2}|F_k|^{1/2}\left[1 + \frac{v_k' F_{*j}^{-1} v_k}{\lambda - 2}^{-(\lambda+p)/2}\right], \\
k &= 1, \ldots, R.
\end{aligned}
$$

This gives the density of $(\alpha_{\mathbf{t+1}}|\mathbf{Y}_t, \psi)$.

Step 3. Produce filtered sample of $\alpha_{\mathbf{t}}$ Resample $\{\alpha_{t+1|t}^{1}, \ldots, \alpha_{t+1|t}^{k}, \ldots, \alpha_{t+1|t}^{R}\}$ M times with probability proportional to

$$\frac{w_k}{w_{k_j}} = \frac{St\{y_t|\alpha_{t+1|t}^{k}, \lambda\}}{St\{y_t|\alpha_{t+1|t}^{k_j}, \lambda\}}.$$

Let the sample indexes be $m_1 \ldots, m_k, \ldots, m_M$, and associate these with $\alpha_{t+1|t}^{m_1}, \ldots, \alpha_{t+1|t}^{m_k}, \ldots \alpha_{t+1|t}^{m_M}$. $m_k = k$, which randomly drawn from $k = 1, \ldots, R$. The draws are viewed as the filtered sample of $\{\alpha_{t+1}^{1}, \ldots, \alpha_{t+1}^{M}\}$ drawn from $(\alpha_{\mathbf{t+1}}|\mathbf{Y}_{t+1}, \psi)$. Each particle is updated as in,

$$
\begin{aligned}
a_{t+1|t+1}^{m} &= a_{t+1|t}^{m_k} + P_{t+1|t}^{m_k}(WZ)'F_{m_k}^{-1}v_{m_k}; \\
P_{t+1|t+1}^{m} &= P_{t+1|t}^{m_k} - P_{t+1|t}^{m_k}Z'W'F_{m_k}^{-1} * W * Z * P_{t+1|t}^{m_k}; \\
m &= 1, \ldots, M.
\end{aligned}
$$

Once we have a sample from $(\alpha_{\mathbf{t+1}}|\mathbf{Y}_{t+1}, \psi)$, the mean of the sample draws provides an estimate of $E(\alpha_{\mathbf{t+1}}|\mathbf{Y}_{t+1}, \psi)$ which is the unbiased estimator of the daily business condition. Furthermore, the particle filtering steps can be used to estimate the likelihood ordinate as summarized by the following steps.

Step 4: Likelihood function of the Model

- Set $t = 1$, initialize ψ and obtain a sample of $\alpha^{(g)}_{t-1|t-1}(g \leq M)$.
- For each value of $\alpha^{(g)}_{t-1|t-1}$ sample

$$\alpha^{(g)}_{t|t-1} \sim N(T\alpha^{(g)}_{t|t}, \sigma^2)$$

- Estimate one-step ahead density as

$$\hat{f}(y_t|\mathbf{Y}_{t-1}, \psi) = \frac{1}{M}\sum_{g=1}^{M} St(\mathbf{Y_t}|\mathbf{Z_t}\alpha_{\mathbf{t}} + \mathbf{\Gamma_t}\omega_{\mathbf{t}}, \mathbf{H_t}, \lambda).$$

- Apply the filtering procedure in Step 2-3 to obtain $\alpha^1_{t|t}, \ldots, \alpha^M_{t|t}$ from $\alpha_{t|t}|\mathbf{Y}_t, \psi$.
- Increment $\alpha_{t|t}$ to $\alpha_{t+1|t}$ and goto 2.
- Return the log likelihood ordinate

$$\log f(\mathbf{y}|\psi) = \sum_{t=1}^{T} \log \hat{f}(y_t|\mathbf{Y}_{t-1}, \psi).$$

The parameters are obtained using maximum likelihood. The details on the constraints, the choice optimization algorithm,and the computation method are discussed in the following section.

4 Empirical Results

4.1 Parameter Estimation

This section discusses the parameter estimation method and reports the estimation results. The parameters are restricted as $-1 \leq \beta_3$, β_4, $\rho \leq 1$, $0 < \sigma^2 \leq 25$ and $2.1 \leq \lambda \leq 30$ to maximize the sample log-likelihood. I convert the constrained optimization into the unconstrained optimization, and finds the optimum using the interior-point algorithm. In addition, the computational speed matters in our context - the computation must be finished within one business day; otherwise, the daily indicator will never be able to assimilate the newly available observation. Traditional computation technique are too slow to to deal with computationally heavy project using particle filter. To increase the computational speed, the device used in this paper is a graphics processing unit (GPU), which consists of many cores (processing units) connected to a host (conventional desktop or laptop computer with one or a few central processing units) through a bus (for data transfer between host and device). The device is designed for single instruction multiple data (SIMD) processing, which is compatible with the setting of particle filter. Besides Matlab, the software used is the CUDA extension of the C programming language (Nvidia, 2011), which is freely available.

Table 1 reports the statistical inference results of the parameters estimated by particle filter. Two models are taken into account: the Gaussian Kalman-filter model (ADS index) and the fat-tailed Particle filter model ($2.1 < \lambda \leq 30$). All parameters are positive and significant at 5% significance level. The fat-tailed Particle filter model estimates the degree of freedom of the Student's t distribution at 3.0465. The small λ strongly rejects the normality of the observable variables. Without taking into account the fat-tailed specification, the Gaussian Kalman-filter Model squeezes all extremal values into variance estimates. $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma$ estimated by the Gaussian Kalman-filter model are larger than those estimated by the fat-tailed Particle filter model[3]. The variance of the shocks to the term spread σ_1 reaches 2.9583 when estimated by the Gaussian Kalman Filter model, which doubles what estimated by the fat-tailed Particle filter model, $\sigma_1 = 1.3370$. The variance of the term spread is $\sigma_1\lambda/(\lambda-2) = 3.8922$ based on the fat-tailed specification. Other parameters estimated by two models are close to one another.

To test whether the non-Gaussian specification is necessary in extracting daily business condition, a likelihood ratio test is considered. The test statistic takes the form of

$$\begin{aligned} D &= -2\ln\left(\frac{\textit{Likelihood for null model}}{\textit{Likelihood for alternative model}}\right) \\ &= -2\underbrace{\ln(\textit{Likelihood for Gaussian Model})}_{null} + 2\underbrace{\ln(\textit{Likelihood for Fat-tailed Model})}_{alternative} \end{aligned}$$

The limiting distribution of D is approximately a χ^2 distribution with degrees of freedom 1[4]. Table 2 reports the likelihood ratio test results. A positive and significant test statistic indicates that we couldn't reject the non-Gaussian state-space model, which better captures the fat-tailed dynamics of the real economy.

4.2 Results Interpretation

Designed to track real business conditions on a daily basis, the extracted business condition index is in accord with the U.S. economic history from 1960 to 2007. It blends high- and low-frequency information and incorporates the dynamics of both stock and flow variables. Its underlying (seasonally adjusted) economic indicators are: the daily yield curve term spread, defined as the difference between 10-year and 3-month U.S. Treasury yields; weekly

[3]The variance of observed variables in the fat-tailed Particle filter model is $\sigma_1\lambda/(\lambda-2), \sigma_2\lambda/(\lambda-2), \sigma_3\lambda/(\lambda-2), \sigma_4\lambda/(\lambda-2)$.

[4]The Gaussian state-space model needs 3 parameters to estimate the variance of $\epsilon_{\mathbf{t}}$, in addition to which the non-Gaussian model uses another parameter λ to control the shape of the Student's distribution. The Gaussian state-space model (null) nests the Non-Gaussian state-space model (alternative).

initial claims for unemployment insurance; monthly employees on nonagricultural payrolls; and quarterly real GDP. Except that monthly payroll employment is stock variable, all three other variables are flow variables. The sample period is from April 1, 1960 to February 20, 2007.

The average value of the ADS index is zero. Progressively bigger positive values indicate progressively better-than-average economic conditions, and vice-versa. The ADS index is used to compare business conditions at different times[5]. A value of -7.0, for example, would indicate business conditions significantly worse than at any time in either the $1990-91$ or the 2001 recession, during which the ADS index never dropped below -5.0. Figure 1 compares the dynamics of the fat-tailed ADS index and Gaussian ADS index. Both ADS indices capture the dynamics of the U.S. economy from 1960 to 2007. The fat-tailed ADS index exhibits two characteristics.

First, the dynamics of fat-tailed ADS index are generally similar to those of the Gaussian ADS index.

The 1960s began with a mild recession. This corresponds to the barely negative start to both ADS indices. The first downward movement appears when unemployment peaked at 7.1% in May 1961. The positive movement in the mid-1960s corresponds to the recovery lead by the Fed's interest rate reduce, as well as the increase in the aggregate expenditure cause by "Kennedy tax cuts" (1964), the "Great Society" domestic programs (the Johnson administration), and the Vietnam War.

The 1970s was a dark period of the U.S. economy. Both ADS indices are highly volatile during this period. The positive start of both indices refer to the economic boom caused by the price control of the Nixon administration and the Fed's low interest rates. The downward

[5]published by the Real-Time Data Research Center of the Federal Reserve Bank of Philadelphia (`http://www.phil.frb.org/research-and-data/real-time-center/`).)

movement of ADS indices started in 1973, when price controls were relaxed and the first oil crisis occurred. In the meantime, there was another adverse supply shock: a sharp increase in food prices caused by bad weather. Inflation rose to 11% in 1974. Then in 1979, the second oil shock struck. Inflation was 11.4% in 1979 and 13.5% in 1980.

The first half of the 1980s is a classic example of a disinflation - a temporary rise in interest rates reduced inflation permanently, at a cost of recession. As Volcker's Fed raised interest rates sharply in late 1979 and early 1980, both ADS indices start with a sharp drop in the 1980s. The positive jump of both ADS indices in 1986 corresponds to the beneficial supply shock - the oil price decrease in 1986. At the end of the decade, strong spending on consumption and investment pushed output above potential and reduced unemployment to 5%. Inflation crept above 5% and the Fed responded by raising interest rates in 1989. This made both ADS indices fall below 0 again in the late-1980s.

The negative ADS indices indicate that the 1990s started with a mild recession, caused partly by the 1989 interest rate increase. Two other factors were negative expenditure shocks: the fall in consumer confidence when Iraq invaded Kuwait, and the credit crunch that followed the $S\&L$ crisis. Both ADS indices enter the 2000s with negative values. This is consistent with that the economy slowed in early 2000s, dampening the euphoria about the New Economy. Responding to the recession, the Fed reduced interest rates steadily. The federal funds rate fell from 6.5% in 2000 to 1.0 percent in late 2003. Low interest rates stimulated spending; Tax-cuts proposed by the new Bush administration (enacted in 2001 and 2002) also stimulate spending. By the end of 2003, unemployment was falling. Both ADS indices exhibit significant positive jump. As the economy recovered, fears of deflation waned and the Fed raised interest rates. In 2005-2006, the economy seemed to settle into an equilibrium, with unemployment around 5%, inflation around 2% and the federal funds rate 5.25%; but both ADS indices have fell below 0, preluding the large crisis starting in August, 2007.

The fat-tailed ADS index better captures the volatile movements of real economy than the Gaussian ADS index.

The fat-tailed ADS index is more sensitive to the volatile movements of the real economy. Four stylized periods are the mid-1970s, the mid-1980s, the mid-1990s, and the beginning of the 2000s. The gaussian ADS index is more robust to the volatile movements of the real economy, and hence is too smooth to reflect sufficient details.

In the mid-1970s, three adverse shocks hit the economy: the relaxed price control, the oil crisis, and food price inflation caused by bad weather. The fat-tailed ADS index exhibits downward movements during this time whereas the Gaussian ADS index does not. During this period, the Fed raised interest rate sharply. High interest rates caused a recession: output fell and the unemployment rate rose to 9.0 percent in mid-1975. As the Fed switched gears and lowered interest rates, both ADS indices moves above zero, but the fat-tailed ADS index moves further.

In mid-1981, the nominal federal funds rate exceeded 17% and the real rate was almost 8%[6]. The decrease in aggregate consumption; along with high interest rates, pushed output down and unemployment up. The spike in interest rates caused the deepest recession since the Great Depression: unemployment reached 10.8% in 1982. This is reflected by the negative drop of the fat-tailed ADS index, but not by the Gaussian ADS index.

Starting in the mid-1990s, the economy exceeded expectations. Unemployment fell below 6% percent in 1994 and reached 3.8% in April 2000. Without spurring inflation, the natural rate of unemployment fell because of high productivity growth. The fat-tailed ADS index displays a lot of positive movements during this period of time whereas the Gaussian ADS index simply remains negative and does not respond to these underlying changes.

The fat-tailed ADS index also reflects the recession started in 2001, caused by the fall in investment when companies scaled back their computer spending. Consumption also fell because the stock market fell, reducing people's wealth. The Gaussian ADS index simply

[6] Around the same time, the Carter administration temporarily imposed "credit controls"

smoothed all these negative shocks and remains positive during this period of time, whereas unemployment peaked at 6.4% in June 2003[7].

5 Conclusion

This paper contributes to ADS index study proposed by Aruoba et al. (2009) by incorporating a fat-tailed distribution of macroeconomic shocks. It provides a more general framework for extracting the business conditions from data observed at fixed frequencies. The computation uses the latest development in graphical computing units (GPU). The extracted business condition index behaves similarly to the ADS index, but can better capture some extreme economic fluctuations.

[7] One possible explanation is that inflation was low entering the 2000s, and the recession pushed it lower. Inflation was about 1% in 2003.

6 Appendix

Table 1: Parameter Estimation Results

(Sample Periods: Apr 1, 1962- Feb 20, 2007)								
	Fat-Tailed ADS Index				**Gaussian ADS Index**			
Parameters	**Coef.**	**Std. Err.**	**t**	**p-value**	**Coef.**	**Std. Err.**	**t**	**p-value**
β_1	0.4543	0.0032	139.46	(0.00*)	0.5753	0.1614	3.56	(0.00*)
β_2	0.5273	0.0032	161.85	(0.00*)	0.6184	0.1663	3.72	(0.00*)
β_3	1.2354	0.0040	306.96	(0.00*)	1.3018	0.2096	6.21	(0.00*)
β_4	1.3670	0.0044	306.96	(0.00*)	1.2327	0.2052	6.01	(0.00*)
γ_2	0.6823	0.0033	209.44	(0.00*)	0.6354	0.1664	3.82	(0.00*)
γ_3	0.6648	0.0032	204.09	(0.00*)	0.7085	0.1654	4.28	(0.00*)
γ_4	0.5976	0.0033	183.46	(0.00*)	0.6486	0.1664	3.90	(0.00*)
ρ	0.5815	0.0021	276.22	(0.00*)	0.6600	0.2059	3.20	(0.00*)
σ_1	1.3370	0.0058	229.58	(0.00*)	2.9583	1.0682	2.77	(0.00*)
σ_2	1.1360	0.0042	270.210	(0.00*)	1.4721	0.3146	4.70	(0.00*)
σ_3	1.0529	0.0036	291.53	(0.00*)	1.1954	0.1593	7.50	(0.00*)
σ_4	1.1283	0.0041	272.04	(0.00*)	1.5195	0.3475	4.37	(0.00*)
σ	1.0706	0.0037	286.71	(0.00*)	1.1579	0.1588	7.29	(0.00*)
λ	3.0465	0.0099	306.96	(0.00*)	$--$	$--$	$--$	$--$

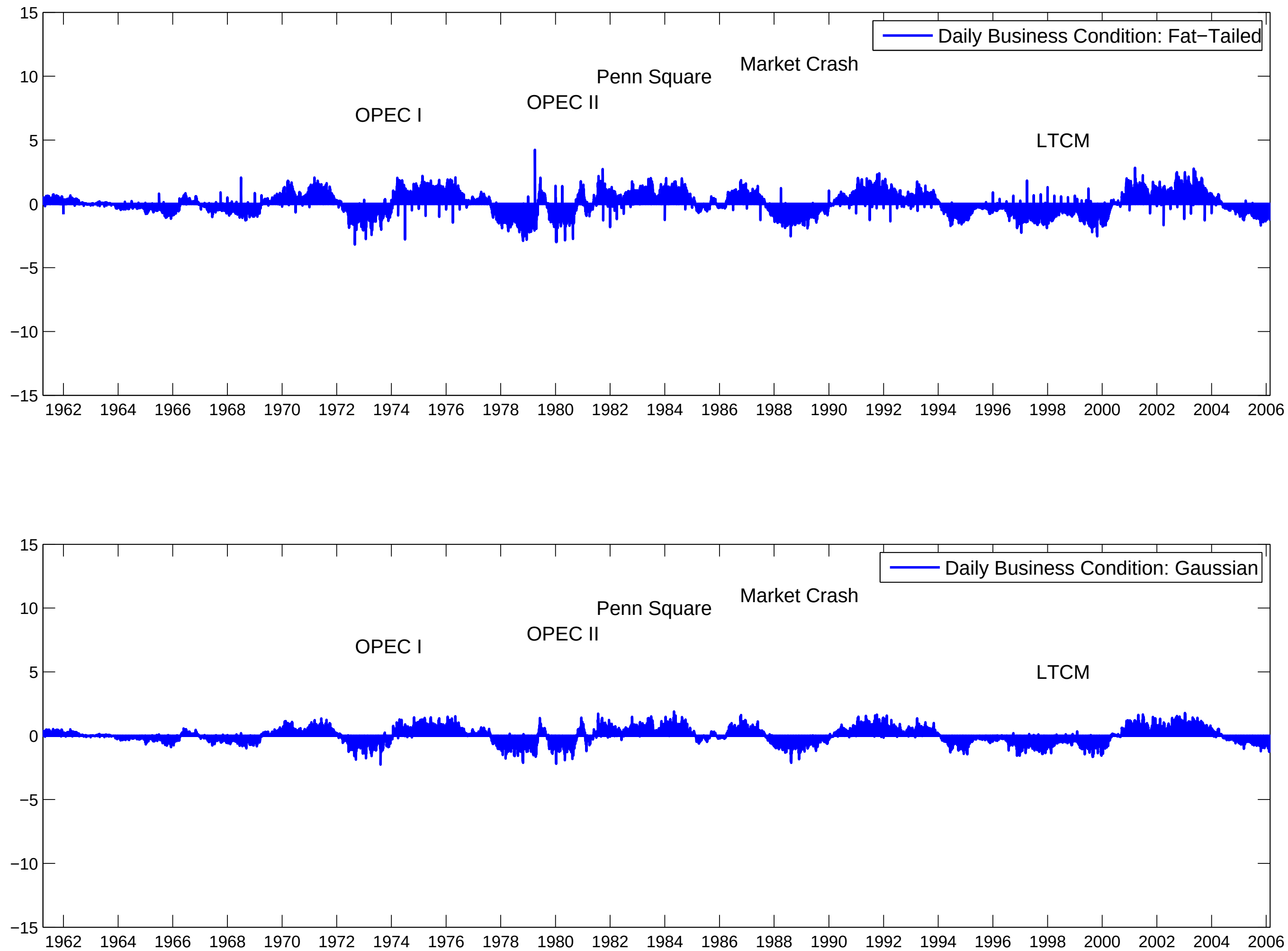
Note: Both state-space models have positive and significant parameter estimates, but fat-tailed state-space model has higher predictive likelihood. The variance of observed variables in the fat-tailed model is $\lambda/(\lambda-2)$ times those of the Gaussian model.

Table 2: Likelihood Ratio Test Results

(Sample Periods: Apr 1, 1962- Feb 20, 2007)		
Model	**Fat-Tailed ADS Index**	**Gaussian ADS Index**
Log-likelihood	-22913.18	-28898.04
Likelihood ratio test statistic	11969.72	
Likelihood ratio test p-value	(0.00*)	

Note: A positive and significant likelihood ratio test statistic indicates that the fat-tailed state-space model significantly outperforms the Gaussian state-space model.

Figure 1: Daily Business Condition Index: Fat-tail vs. Gaussian



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