Real Time Density Forecasts of Output and Inflation via Quantile Regression

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Abstract

This paper compares the density forecasting performance of quantile regressions in real-time with that of two conventional density forecast methods, using U.S. inflation and output growth data from 1959 to 2010. Both forecasts with and without factors are included, and in the latter case, factor selection rules are compared. The density forecasts using quantile regressions are found to be significantly more accurate than the two conventional density forecasts. Including factors extracted from macroeconomic indicators in the quantile density forecast also generally improves density forecast accuracy.

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Contents

1 Introduction

Density forecasting has been gaining increasing attention in the last decade. It represents a complete characterization of the uncertainty associated with a forecast, and thus informs the user of the forecast about the risks involved in using the forecast for making decisions. This paper considers density forecasts of U.S. inflation and economic growth, through which I compare the performance of three commonly used density forecasting approaches. At the simplest level, the variable of interest is assumed to be normally distributed, in which case, all one needs to predict are the conditional mean and the conditional variance. The second approach also centers at the conditional mean, but to avoid parametric assumption about the error term, it instead uses non-parametric kernel methods to develop the density forecast. The third approach, which has gained prominence recently, is a quantile regression. As the ordinary linear regression and the maximum likelihood fit the conditional mean of a time series, a quantile regression instead fits the conditional quantiles. Connecting all the conditional quantile estimates forms an estimation of the empirical cumulative distribution function of the variable to be forecasted.

Meanwhile, another recent development in the forecasting literature has been the recognition that combining the information in large datasets in a way that avoids estimation of too many parameters may be helpful for forecasting. An approach that has been found to be useful for point forecasting is to consider a factor-augmented autoregression, i.e. an autoregression augmented by the first few principal components from a large dataset (see for example Stock and Watson (1998), Stock and Watson (2003), Stock and Watson (2005), and Bernanke et al. (2005)). It would be interesting to know whether such an approach is helpful in the density forecasting context as well. Therefore, in applying each density forecasting approach above, I consider univariate autoregressive models and factor augmented autoregressions using principal components from a large dataset with about 100 series. In addition, to see whether the model selection rules designed for factor-augmented point forecasting still work well in density forecasting, various approaches to select predictors following Bai and Ng (2008) are taken into account.

The main contribution of this paper is to find that density forecasts using quantile regression are significantly more accurate than those using two conventional conditional-mean-centered density forecasting methods: One assumes forecast errors are Gaussian and imposes normality on the true data generating process. The other allows forecast errors to be non-Gaussian and attaches the nonparametric kernel density of forecast errors to the conditional mean forecasts. I do an out-of-sample comparison of the quantile forecasts relative to the two simple benchmarks using real-time U.S. data from January 1959 to June 2010 at both monthly and quarterly frequencies. The density forecasts being compared are both autoregression and factor augmented autoregression. The comparison uses the test statistic proposed by Amisano and Giacomini (2007) but takes into account the distortion in the asymptotic distribution when the models being compared are nested.

Some authors have considered quantile density forecasts for economics activity and inflation, such as Manzan and Zerom (2009), Gaglianone and Lima (2009) and De Nicolo and Marcella (2010), but none of these papers uses factors from a large dataset for prediction. Also, Manzan and Zerom (2009) related first differences of inflation to predictors, imposing a non-stationary process to inflation dynamics. Moreover, the current paper studies the density forecasting performance of various factor selection methods, because no attention has been paid to whether model selection rules designed for point forecasting can work well in density forecasting or not. It thus especially contributes to the multivariate density forecasting work using quantile regression in the sense of choosing powerful predictors.

This paper yields the following results. Firstly, density forecasts using quantile regressions are significantly more accurate than the two conditional mean-centered density forecasts. This is true across forecasting horizons and both in autoregressions and factor-augmented autoregressions. Secondly, including factors in the quantile density forecast generally improves forecast accuracy, but including too many factors may cause the accuracy to deteriorate. Thirdly, the best way of selecting predictors for density forecasting is the Least Angle Regression (LARS). Other predictor selection rules, like the hard threshold, do not work as well in density forecasting.

The reminder of the paper proceeds in five sections. Section 2 lays out the three density forecast methods. Within each method, I compare the performance of autoregressions and factor-augmented autoregressions using four different possible factor selection methods. Section 3 introduces two sets of tests used in the comparing the performance of density forecasts across methods and models. It also discusses how the distribution of these test statistics are affected when the methods being compared are nested. Section 4 discusses the empirical results from comparing the density forecasts across methods and models. Section 5 concludes.

2 Density Forecasting: Methods and Models

2.1 Methods: Mean-centered versus Quantile-based

In this paper, I consider three methods for density forecasting:

The first method involves a linear regression with normal forecast errors. This consists of an ordinary least square regression centered at the conditional mean and assuming forecast errors are Gaussian zero-mean to imply a density forecast.

$$f(Y_{t+h}^h|X_t) = \phi\Big(\frac{Y_{t+h}^h - \widehat{Y}_{t+h}^h}{\hat{\sigma}_y}\Big). \tag{1}$$

$\phi(\cdot)$ is the probabilistic density function of the standard normal distribution. $\hat{Y}^h$ is the conditional mean forecast estimated from the ordinary least square regression: $\hat{Y}^h = \widehat{\theta^h} X_t$.

$\widehat{\theta}^h$ and the conditional variance $\hat{\sigma}_y^2$ are estimated from the regression:

$$Y_t^h = \theta^h X_{t-h} + \varepsilon_t, \qquad \varepsilon_t \sim_{i.i.d} N(0, \sigma_y^2).$$

With Gaussian innovations, the variable of interest is normally distributed. The method is so-called **normal density forecast**. To predict the entire distribution, all one needs to know is the parameters for the first two moments - conditional mean and conditional variance.

Although it has the advantage of being simple, the method is rigid. Forecast errors are not always Gaussian. The second method starts with the same conditional mean-centered regression but uses the kernel density of the forecast errors. In particular, the method assumes all possible outcomes of Y_t^h at time t can be obtained by attaching all historical forecast errors to the conditional mean forecast,

$$Y_{t+h}^h = \widehat{Y}_{t+h}^h + e_s, \qquad \text{for } s = 1, \ldots, t.$$

The density forecast then is given by the kernel density of Y_t^h using these fabricated possible outcomes.

$$f(y_{t+h}^h | X_t) = K\big(\frac{y_{t+h}^h - Y_{t+h}^h}{h_n}\big). \tag{2}$$

where Y_{t+h}^h is the sum of the mean forecast made at t plus all historical forecast errors. $K(\cdot)$ is the kernel density function, which typically entails the choice of bandwidth h_n. $f(Y_{t+h}^h | X_t)$ is the predictive density estimated at the realization Y_{t+h}^h. The method is later referred as **semi-parametric density forecast** for it estimates the conditional mean via a parametric method-OLS but the entire distribution using nonparametric kernel-smoothing technique.

In both methods above, the conditioning variables X_t affect the density forecast solely through the conditional mean. To allow for more flexibility, the third method considers quantile estimates. The quantile $Q_{\tau_i}(Y_{t+h}^h|X_t)$ is the value such that τ_i (percent) of the mass of the distribution $F(Y_{t+h}^h|X_t)$ is less than $Q_{\tau_i}(Y_{t+h}^h|X_t)$. To obtain a density forecast, it is natural to pick a bunch of τ's: $\tau_1, \ldots, \tau_n$ and then apply the kernel smoothing technique to the corresponding n quantile estimates. Each quantile is predicted by

$$\hat{Q}_{\tau_i}(Y_{t+h}^h|X_t) = \widehat{\theta^h}(\tau_i)X_t.$$

$\hat{\theta}^h(\tau_i)$ is estimated following Koenker and Bassett (1978). It describes how the τ_i-th quantile of the h-period ahead conditional distribution might vary with the regressors in X_t. The conditional distribution obtained in this way can take any shape. The method is hence named **quantile density forecast**. When Y_{t+h}^h is normally distributed, the median forecast $\hat{Q}_{0.5}(Y_{t+h}^h|X_t)$ given by quantile regression equals the conditional mean forecast given by OLS. With all other quantiles encompassed by the corresponding percentiles of the normal distribution, the quantile density forecast is nested by the normal density forecast.

Moreover, the choice of τ's raises a couple of issues. First, the smallest and the largest τ_i have to be chosen sensibly. For the conventional quantile regression does not work well on the "extremal quantiles" (see, for example, Chernozhukov (2005)). Second, since the coefficients in quantile regression are estimated independently, higher quantiles may be smaller than the lower quantiles. The quantile function may exhibit "crossings". To deal with this issue, I follow Chernozhukov et al. (2010)'s approach and rearrange the original quantile estimates into ascending order. The entire conditional distribution can then be estimated using the nonparametric kernel density. Specifically,

$$f(y|X_t) = \frac{1}{nh_n}\sum_{i=1}^{n} K\Big(\frac{y - \hat{Q}_{\tau_i}(Y_{t+h}^h|X_t)}{h_n}\Big), \tag{3}$$

in which y are evenly interpolated points that generates the domain of $f(y|X_t)$. n is the total number of quantiles that are estimated. Epanechnikov kernel is chosen in empirical setting. The estimated kernel density are normalized to satisfy $\int K(u)du = 1$. The predictive density is right the kernel density evaluated at the realization Y^h_{t+h}. This approach was also used by Gaglianone and Lima (2009).

2.2 Models: Autoregression versus Factor Augmentation

In evaluating the forecasting performance of each method above, I considered five different models. The autoregressive model (AR/QAR) serves as the benchmark. Four factor-augmented autoregressive models are also considered. The variable of interest is future h-period growth Y^h_{t-1+h}, given information available at time t. The conditioning variables are past values of Y^h_{t-1+h} and static factors extracted from many macroeconomic indicators selected through various ways. Let k be the index for the four factor-augmented models, which differ in the factors that are included, as described below. The forecasting equation augmented by the k-type factors is given by

$$Y^h_{t-1+h} \quad = \quad \gamma + \sum_{j=0}^{p-1} \alpha_j Y_{t-1-j} + \sum_{i=1}^{r} \beta^k_i F^k_{i,t-1} + \varepsilon^h_{t-1+h}, \tag{4}$$

whereby p, the autoregressive lags and r, the optimal number of factors are selected by the Bayesian information criterion (BIC). The static factors F^k_{t-1} are taken to be the first r principal components of a large number of predictors P^k_{t-1}. The predictors may either be the entire datasets as in Stock and Watson (1998) or be selected through various ways following

Bai and Ng (2008). Five variations of the factor augmentation are:

$$k = \begin{cases} \textbf{AU} & \text{no factor augmentation.} \\ \textbf{PC} & \text{factors are principal components of all predictors.} \\ \textbf{Hard} & \text{factors are principal components of predictors selected by hard threshold.} \\ \textbf{LARS} & \text{factors are principal components of predictors selected by LARS.} \\ \textbf{LARS(SPC)} & \text{uses LARS to select predictors from } [P_{t-1} \;\; P_{t-1}^2]. \end{cases}$$

Details on the hard threshold and LARS predictor selection rules are enclosed in appendix B.

In point forecasting, the principal component method and the hard threshold are found most useful in predicting real economic activity. LARS-based methods are found to have the best performance in forecasting inflation (see Bai and Ng (2008) and Stock and Watson (2009)). In density forecasting, little or no attention has been paid to examine whether the predictor selection rules designed for point forecasting can select predictors that are useful in forecasting the entire distribution. Four factor augmented models ($k = PC, \ldots, LARS(SPC)$) are hence compared with one another as well as with the univariate autoregression models to study their density forecasting performance.

I adopt the recursive method to obtain the out-of-sample forecasts. T is the total number of observations and m is the number of observations used to construct the first forecast. Under the recursive scheme, the parameters are updated as forecast moves forward through time: for $t = m+1, \ldots, T$, fixing the forecasting origin to the first in-sample period $t = 1$. At each time $t = m+1, \ldots, T$, the parameter estimate depends explicitly on all information from $1, \ldots, t$. Using these parameter estimates, conditional distributions and predictive densities are constructed for each model in every period.

Furthermore, since a good density forecast will have the feature that the realized value will take place at a point with a high predictive density, the measure of accuracy for a particular density forecast thus uses the *predictive likelihood*, which is the sum of logarithmic predictive densities over the out-of-sample forecasting periods:

$$h(T) = \sum_{t=m+1}^{T} f_t^k(Y_{t-1+h}^h | X_{t-1}^k).$$

$f_t^k(Y_{t-1+h}^h | X_{t-1}^k)$ is the predictive density of Y_{t-1+h}^h implied by model-k at date t. At the simplest level, one can plot the predictive likelihood against out-of-sample periods to examine how the predictive ability of a particular density forecast model evolves over time.

3 Evaluating the Density Forecasts

The motivation for this paper is to explore whether quantile regression has better performance against the conditional mean-centered methods in forecasting the distribution of future output growth and inflation. This raises the question on how to evaluate a density forecast method. A good density forecast should meet two criteria: First, when subjecting to the probability integral transform, it should be uniform and i.i.d., otherwise, the forecast may not be efficient; Second, among multiple density forecasts that satisfy the uniformity and i.i.d. after transform, it should exhibit higher forecast accuracy than other density forecasts. The evaluation therefore consists of two tests. The first test examines whether there is misspecification in the predictive distribution so that the density forecast is “unbiased”. The second test compares the predictive likelihood of two different density forecasts. The comparison is set between the quantile and the non-quantile forecasts, with other forecasting conditions such as conditioning variables, model selection methods and forecasting scheme, etc. the same in each comparison.

3.1 Test the Goodness of Fit

To assess whether there is distributional misspecification in the quantile density forecast, I use the the "goodness of fit" test developed by Diebold et al. (1998). The idea behind the test is simple: forecasts put through the probability integral transform are uniform and i.i.d.. In particular, let Z_t be the percentile of where the realization at t was observed in the *ex-ante* forecast density. If the density forecast is correctly specified, Z_t should be uniformly distributed on the unit interval and purely random over time. If the density forecast is poorly specified, the empirical CDF of Z_t will violate the unconditional uniformity, or $i.i.d$, or both. The simplest way to construct such a test is to compare the CDF of Z_t with that of a percentile index sliding from 0 to 1, which examines whether the "occurrence percentile" meets the percentile that it is supposed to do. The closer the two CDFs are, the closer the predictive distribution to the true, and the less likely the density forecast is misspecified.

Specifically, Diebold et al. (1998) suggests combining the Kolmogorov-Smirnov (KS) test with graphical tools to test the "goodness of fit". The metric of KS test statistic is simply the maximum distance between the empirical CDF of Z_t and CDF of the percentile index. Under the null hypothesis, the test statistic converges in distribution to the supreme of the absolute value of a Brownian bridge. The acceptance of the KS test signifies a "good fit" of a particular density forecast. Table $3.a$ and $3.b$ report the results of KS-test and show that models using quantile density forecast method generally exhibits a good-fit.

3.2 Compare the Density Forecast Accuracy

To decide whether the quantile density forecast has a better out-of-sample predictive ability, I use the difference-in-predictive-likelihood test proposed by Amisano and Giacomini (2007). The test pairwisely compares the forecasts from the quantile models to those from the mean-centered models with respect to the logarithmic predictive density. The advantage of one density forecast over another is measured by the difference of the logarithmic predictive densities in every period: $\Delta L_t(Y^h_{t-1+h}) = \log f(Y^h_{t-1+h}|X_{t-1}) - \log g(Y^h_{t-1+h}|X_{t-1})$, where $f(\cdot)$ is the predictive density of the quantile density forecast; and $g(\cdot)$ is that of either the conditional mean-centered density forecast. Based upon the sequence of $\{\Delta L_t(Y^h_{t-1+h})\}^T_{t=m+1}$, the test statistic takes the form of a t-statistic:

$$AG_{m,T,h} \equiv \frac{\Delta \overline{L}_t(Y^h_{t-1+h})}{\hat{\sigma}/\sqrt{(T-m)}}, \quad (5)$$

where $\Delta \overline{L}_t(Y^h_{t-1+h})$ is the sample average of $\Delta L_t(Y^h_{t-1+h})$, and $\hat{\sigma}^2$ is a suitable heteroskedasticity and autocorrelation consistent (HAC) estimator of the asymptotic variance σ^2 (see Newey and West (1987)).

$$\hat{\sigma^2} \equiv \frac{1}{T}\sum_{t=m+1}^{T} \Delta L_t^2 + 2 \cdot \frac{1}{T}\sum_{j=1}^{J}(1-\frac{j}{J+1})\sum_{t=m+1+j}^{T} \Delta L_t \Delta L_{t-j}.$$

For one-period ahead forecast ($h = 1$), the choice of lag truncation J is set to be 0 as in Giacomini and White (2006) and Amisano and Giacomini (2007). For longer horizon forecast ($h > 1$), the choice of J follows Andrews (1991).

Based upon the value of this statistic, one either fails to reject or rejects the null of equal predictive ability. The null and alternative can be stated as

$$H_0 : E[\Delta L_t(Y^h_{t-1+h})] = 0 \;\; v.s \;\; H_A : E[\Delta L_t(Y^h_{t-1+h})] > 0.$$

The test is one-sided rather than two sided because the two models are nested. That is, if the linear regression model is correctly specified, then it will be equivalent in population to the quantile regression. It is impossible for the more flexible quantile regression to be less accurate than the linear regression, in population. The null hypothesis will be rejected if model-$f(\cdot)$ (quantile density forecast) provides more accurate density forecasts relatively to model-$g(\cdot)$ (conditional mean-centered density forecasts), in which case, $AG_{m,T,h} > 0$.

As a result of the fact that the two models are the same under the null hypothesis, the asymptotic distribution of the AG statistic is not normal (it would be asymptotically normal if the models were not nested (Amisano and Giacomini (2007))). The issue is familiar in the context of point forecasting. For point forecasting, the analog of the AG statistic is the test proposed by Diebold and Mariano (1995). In the nested case, this has a nonstandard distribution, derived by Clark and McCracken (2001), Clark and McCracken (2005), McCracken (2007), which is a function of stochastic integrals of Brownian motion. But for nested density forecasts, no corresponding results are known.

I consider two approaches to deal with the issue. The first is to pretend that the asymptotic distribution of the AG statistic is still standard normal, and make inference using standard normal critical values[1]. The second is to use a bootstrap approach. Both are reported in empirical results.

As for the *nested point forecasts*, the nonstandard asymptotic distribution of AG test statistic has been derived by Clark and McCracken (2001), Clark and McCracken (2005) and McCracken (2007), which is a functions of integrals of Brownian motion and may or may not depend upon unknown nuisance parameters. As for the *nested density forecasts*, except that it is nonstandard, little is known. In particular, there are two approaches to deal with this issue. The first is to pretend that the asymptotic distribution is still standard normal and make inference using standard normal critical values. For example, one might use the *rolling window approach* as in Giacomini and White (2006) to preserve the finite sample imprecision in parameter estimation so that the difference of predictive likelihood will not vanish asymptotically. On the other hand, in a *recursive forecast scheme*, both the in-sample and the out-of-sample sizes grow, which causes the difference of predictive likelihood to vanish asymptotically. As such practitioners have to turn to bootstrap approaches and Monte Carlo analysis of finite-sample size and power to inference. As a result, the finite-

[1]This would be justified in a rolling window approach as in Giacomini and White (2006) in which the finite sample imprecision in parameter estimation is preserved and so that the difference in predictive likelihood will not vanish asymptotically

sample distribution of AG statistic in recursive density forecast bears a strong resemblance to those in Clark and McCracken (2005) and McCracken (2007).

3.2.1 Bootstrap Algorithm

The bootstrap aims to simulate finite-sample distribution of AG statistic under the null using the actual sample. The restricted model is the normal density forecast with Gaussian errors; the unrestricted model is quantile density forecasts. The null that the additional quantiles does not provide predictive content over the mean and the variance requires us to simulate Gaussian forecast errors for bootstrap samples. This makes the bootstrap method used here differ from the parametric bootstrap method used by Kilian (1997) and Stock and Watson (2003). The stored residuals are not directly taken for sampling, but their standard deviation is taken to simulate normally distributed forecast errors. The Gaussian forecast errors are then obtained by sampling from these normally distributed artificial residuals.

Bootstrapped time series on Y^h_{t-1+h} and F_{t-1} are parameterized using model estimates based on monthly (quarterly) $1959:01-2010:06$ data - only the last data vintage. The Y^h_{t-1+h} takes exactly the same form as the normal density forecasting model for $h=1$. The F_{t-1} follows an vector autoregression for r static factors. The lag orders for Y_{t-1} and F_{t-1} are fixed to be one. Forecast errors are simulated by drawing with replacement from the Gaussian forecast errors. The first observation is drawn from the actual data. More details about the bootstrap DGP are enclosed in appendix A.

In each of 400 bootstrap replications, the bootstrapped data are used to recursively estimate the conditional mean-centered density forecast (restricted) and the quantile density forecast (unrestricted). The estimated models for Y^h_{t-1+h} and F_{t-1} are taken to be correctly specified in bootstrapping artificial data. The resulting forecasts are used to calculate forecast test statistics. Critical values are simply computed as percentiles of the bootstrapped test statistics.

3.2.2 Monte Carlo Evidence

It is important to provide Monte-Carlo simulation evidence on the finite sample size and power of the AG test. Ideally, I would do this for both the version of the test using normal critical values and the version of the test using bootstrap critical values. Unfortunately, the computation cost of doing a bootstrap evaluation of out-of-sample forecasts within a Monte Carlo simulation is too great. For the bootstrap, I rely on the fact that the bootstrap often works well in small samples, though acknowledge that this is by no means always the case. Thus, my Monte Carlo simulations consider only the size and power of the AG test comparing the sample test statistics to standard normal critical values (500 replications per simulation).

The Monte-Carlo DGP consists of one power DGP and two size DGPs - one without factors and the other in presence of factors. Each experiment is performed at both monthly and quarterly frequency. The sample length is taken to be the same as those in the real data. The number of replications for each experiment is 500. For both size and power DGPs, I generate data using independent draws of innovations from the normal distribution and the autoregressive structure of the DGP. The initial observation of each DGP uses the unconditional mean of each simulated data series. I consider results for a variety of forecast frequencies: As for monthly forecast, Y is the monthly CPI with forecast horizons: $h =$ 1, 3, 6, 12 months. The DGP is parameterized using model estimates based on monthly $1959:01-2010:06$ data. The total sample length is $T = 622$ months with the first recursive in-sample periods $m = 120$ months; As for quarterly forecast, Y is the quarterly GDP with forecast horizons: $h = 1$, 2, 4, 8 quarters. The DGP is parameterized using model estimates based on quarterly $1959:Q1-2010:Q2$ data. The total sample length is $T = 210$ quarters with the first recursive in-sample periods $m = 40$ quarters.

Size DGP To evaluate the actual size of the AG test when models in compare are nested, the DGP ensures that the restricted model (mean-centered density forecasts) are true in real-time. The data generating process follows:

$$\begin{bmatrix} y_t \\ f_t \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & \phi \end{bmatrix} \begin{bmatrix} y_{t-1} \\ f_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_y \cdot u_{y,t} \\ u_{f,t} \end{bmatrix}, \tag{6}$$

with a constant γ, where $\begin{bmatrix} u_{y,t} & u_{f,t} \end{bmatrix}' \sim_{i.i.d} N(0_{(1+r)\times 1}, I_{(1+r)})$. Without factor augmentation, the coefficient β is set at 0. In presence of factor augmentation, factor is simulated through an univariate $AR(1)$ process, with autoregressive coefficient ϕ set at 0.75. The predictors X_t are generated by $X_t = \Lambda f_t + \epsilon_{X_t} \sim N(0,1)$. Parameters α, β, Λ and σ_y were all calibrated from the real data.

Power DGP To evaluate the actual power of AG test, the DGP ensures the quantile density forecast outperforms the mean-centered density forecasts in real-time. This is achieved by specifying that

$$y_t = \gamma + \alpha(U_t) y_{t-1} + u_{y,t}, \tag{7}$$

where $u_{y,t}$ is standard normal, U_t is uniformly distributed on the unit interval, and the autoregressive coefficient α is selected by U_t in way that

$$\alpha(U_t) = \begin{cases} 0 & \text{if } U_t < 0.5; \\ c & \text{if } U_t \geq 0.5. \end{cases} \tag{8}$$

U_t corresponds quantile τ in models. The CDF of y_t takes the form:

$$F_y(\tau) = \begin{cases} \gamma + \Phi^{-1}(\tau), & \text{if } \tau < 0.5; \\ \gamma + c y_{t-1} + \Phi^{-1}(\tau), & \text{if } \tau \geq 0.5. \end{cases} \tag{9}$$

c is set to slide from 0.1 to 0.8. Φ^{-1} is the inverse CDF of the standard normal distribution and so the lower regression model is normally specified. As the autoregressive coefficient stochastically switches between 0 and c, the data series stochastically switches between a random walk and a Gaussian $AR(1)$. The constant γ was set at the fitted values from an autoregression using the real data. The power experiment is not factor-augmented.

Each Monte Carlo simulation involves first estimating restricted and unrestricted models using the density forecast methods introduced in Section 2. The statistics computed with the Monte Carlo data from a given draw represent the "sample" statistics. Based on 500 Monte Carlo draws, I report the percentage of Monte Carlo trials in which the null of equal predictive accuracy is rejected in favor of quantile density forecast. It is the percentage of trials in which the sample test statistics exceed the standard normal critical values. In the reported results, the tests are compared against 5% one-sided standard normal critical values, so that the nominal size of the tests is 5%. Using 10% critical values yields similar findings.

Monte Carlo Results Table $4.a - 4.b$ and Table 5 provide the actual size and power of the AG test, respectively. The results presented in Table $4.a - 4.b$ and Table 5 indicate following main findings:

First, the AG test is undersized at all forecasting horizons. For example, as shown in Table $4.a$'s results for the DGP of a univariate autoregression, with $T = 210$ and $m = 40$, the 5% size of the AG test ranges from 0.00 ($h < 8$) to 0.04 ($h = 8$); 10% size ranges from 0.00 to 0.11 ($h = 8$). As the sample length increases from $T = 210$ to $T = 622$, the 5% size all decrease to 0.00; the largest 10% size decreases to just 0.02 ($h = 12$). Figure $1.a - 1.b$ and $2.a - 2.b$ show the finite-sample distribution of AG statistic is skewed to the right with respect to the standard normal distribution when two models in compare are nested. Figure

$1.a - 1.b$ and $2.a - 2.b$ also indicate that the AG statistics for large samples have heavier tails and drift into the negative orthant much quicker than for small samples. The size distortion establishes the findings by bootstrap. The under-size property implies that the standard normal critical value might signal that the quantile density forecast is not superior in cases where the quantile density forecast is superior.

Meanwhile, the AG test has very low power for $c < 0.3$, and becomes more powerful as c increases, as forecast horizon shortens, or as sample size increases. For example, in Table 5, for one-step ahead autoregressive forecast, as c increases from 0.1 to 0.8, the power increases from 0.00 to 1.00; Setting $c = 0.7$, as h increases, the power decreases from 0.99 to 0.08 in quarterly forecast, and from 1.00 to 0.10 in monthly forecast; As sample size increases from $T = 210$ to $T = 622$, the power almost doubles at medium-long horizons when the comparison is set between quantile density forecast and normal density forecast, but decreases when the comparison is set between quantile density forecast and semi-parametric density forecast. The one-step ahead forecasts are often more powerful than forecasts at other horizons. Forecasts at longer horizons are subject to temporal averaging problem. The underpower property indicates that the standard normal critical value might signal that the quantile density forecast is not superior in cases where it is superior.

Overall, although assimilating the impacts of the semi- and non-parametric density forecasts into the asymptotic distribution of AG statistic requires non-trivial calculations, density forecast users can rely on the finite-sample distributions to make inference. The nonstandard limiting distribution of the AG statistic for nested density forecast models can be well-approximated by the finite-sample distributions given by the bootstrap approaches or the Monte Carlo analysis.

4 Empirical Results

4.1 Data and Methods

This section presents results on the application of the methods, models and tests introduced in Section 2 and Section 3. I focus on forecasting two macroeconomic variables for the United States: inflation, as measured by the log growth of consumer price index (urban, all items) (CPI), and real economic growth, as measured by the domestic output growth (GDP). Data on CPI and GDP are taken from the Federal Reserve Bank of Philadelphia's Real-Time Data Set for Macroeconomists (RTDSM). The data set reflects what the macroeconomic data exactly looked like at each historical date so that forecasting using this data set unveils what the forecast would have performed as they looked at the time (Clark and McCracken (2009)). Let z_{t-1} be the observation at each date t: all forecasts are made using annualized $Y^h_{t-1+h} = \frac{1}{h}\log\left(z_{t-1+h}/z_t\right)$ and $Y_{t-1} = \log\left(z_{t-1}/z_{t-2}\right)$. As for the predictors, I use a 68-variable dataset directly gathered from the Federal Reserve Bank of Saint Louis database FRED[2] than using the 215 series of fully-revised data used by Stock and Watson (2002a). In addition to the reason of data availability, the relatively parsimonious dataset is chosen because Faust and Wright (2009) and Bernanke and Boivin (2003) both find that factor models generally give less accurate point forecast in the smaller dataset than in the 215-variable dataset. These series were all transformed to be stationary by taking first or second differences, logarithms, or first or second differences of logarithms, following the standard practice. Forecasting is executed at both monthly and quarterly frequency. The sample period spans from January 1959 to June 2010.

For the estimation, I use a recursive scheme in the pseudo-out-of-sample forecasting. This entails fully recursive factor extraction, model selection, and density estimation. For

[2] I would like to thank Jonathan for always bailing me out on dataset throughout the days I was working on this paper.

example, for $h = 1$, the first forecast was made using data from 1959 : 01 through 1968 : 12, forming a predictive distribution for Y^h_{t-1+h} of 1969 : 01. As the realization value of 1969 : 01 comes in, all factors and information criteria were then recomputed using data from 1959 : 01 through 1969 : 01, and form a predictive distribution for Y^h_{t-1+h} of 1969 : 02. All order and predictor selection is fully recursive. The conditioning variables used to produce density forecasts change from one month to next. What is constant is only the rule by which that model is selected. Moreover, monthly GDP data is simulated in real-time via Chow-Lin interpolation using six monthly macroeconomic series[3].

4.2 Forecasting Results

Using the above methodology, Figure 5.b, as an example, shows how the out-of-sample predictive distribution of the quarterly output growth from January 1960 to June 2010 evolves using quantile autoregression. The drastic distributional change in the mid-1980s justifies the finding by McConnell and Perez-Quiros (2000) and Kim and Nelson (1999) that there is a substantive decline in the volatility of output growth since 1984.

Tables 1.a − 1.d and Tables 2.a − 2.d report the out of sample forecast results. The forecasting horizon is $h = 1, 3, 6, 12$ for monthly forecasts, and $h = 1, 2, 4, 8$ for quarterly forecasts. In each table, the comparison is two-fold: between quantile and mean-centered methods and among different sets of predictors. The comparison among methods provides evidence that supports the advantage of using quantile regression over conditional mean-centered methods. The comparison among predictors explores the role that dynamic factors play in forecasting the conditional distribution of future inflation and output growth. To avoid clutter in notation, *CPI-Hard Threshold* refers density forecast for future inflation in

[3]The six series are: civilian employment, all items consumer price index for all urban consumers, real disposable personal income, industrial production index, ISM Manufacturing: PMI composite index, all employees total non-farm payrolls, and real personal consumption expenditures. They are all the seasonally adjusted annual rate of the monthly data series.

which the predictors are pre-selected by the hard threshold; *GDP-QAR/AR* refers density forecast for future output growth conditioning on the autoregressive lags; and so on. In each model, entries in the first row are predictive likelihood, in which a larger value signifies a more accurate density forecast. Entries in the second row are the AG test statistics. When the AG test statistic is positive and significant, it indicates that the quantile density forecast provide more accurate density forecasts than normal density forecasts (first column) or than semi-parametric density forecasts (second column). The third row reports the bootstrap p-values. The fourth row reports asymptotic p-values, by which I mean the p-values obtained by comparing the sample test statistic to the standard normal distribution. The conclusions for inflation and output growth are different in details but share three common characteristics.

First, density forecasts using quantile regression are significantly more accurate than the two conditional mean-centered density forecasts. This is true across forecasting horizons and both in autoregressions and factor-augmented autoregressions.

As indicated by Tables $1.a - 1.d$ and Tables $2.a - 2.d$, when the comparison is between quantile density forecast and normal density forecast, the AG statistics are all positive and significant of the 5% or 10% level at all forecasting horizons. The significance holds even when the inference uses (potentially conservative) standard normal critical values. When the comparison is between quantile density forecast and semi-parametric density forecast, the results for inflation and output growth are slightly different. For inflation, there is significant improvement from using the quantile regression at all forecasting horizons in univariate forecast but only at short horizons in presence of factors. For output growth, the AG test is not always significant and there is no certain pattern associated with the modeling specification or the forecasting horizon. In addition, holding other forecasting condition the same, Figures $6.a - 6.b$, as an example, plot how the forecasting accuracy (in terms of predictive likelihood) of the quantile density forecast exceeds that of the normal density forecast as the out-of-sample periods increase.

Second, including factors in quantile density forecast generally improves forecast accuracy but including too many factors may cause the accuracy to deteriorate.[4]

Figures $4.1.a - 4.3.b$ plot the predictive likelihood against the number of factors for all the predictor selection methods that I consider. The multivariate factor-augmented models outperform the autoregression model if the number of factors is small. Except in semi-parametric forecast of inflation, one can always observe a decline in the predictive likelihood as more factors are included. Even with semi-parametric density forecasts where factor augmentation seems crucial in improving forecast accuracy, including more than ten factors does not show much advantage over the forecast using only the first two factors.

For inflation, factors are very useful in predicting the conditional distribution of future inflation, especially when using quantile regression. For output growth, only a small number of factors have strong predictive power over the autoregressive lags. This is true across forecasting approaches and horizons. Univariate autoregressive density forecast of quarterly output growth shows remarkable advantage at medium and long horizons. This indicates that fluctuation in macroeconomic indicators are more relevant in predicting the short-run fluctuation in the distribution but not the long run trend of output growth.

Third, the best way of selecting predictors for density forecasting is based upon Least Angle Regression (LARS). Linear factor selection rules such as hard threshold and Bayesian information criterion do not work well in density forecasting.

[4] Taking into account the temporal averaging issue, the analysis is mainly base on the results from the forecasting of monthly CPI and that of quarterly GDP.

Least Angle Regression outperforms other selection methods for two reasons: First, the principal components are the linear combination of the predictors that explain most variances in the predictors, but do not consider their relationship with the variable being forecast. LARS is a variable search algorithm that mitigates this problem. Second, hard thresholding selects predictors of high marginal predictive power, but the selected predictors might be correlated with one another, which turns out to impair the density forecast. LARS searches predictors that are most correlated with the variable of interest but least correlated with the previous selected predictors. In addition, LARS(SPC) goes one step further and uses the squared predictors to incorporate higher-moment relationships between the variable to be forecast and the predictors. In Figure $4.1.a - 4.3.b$, as the number of factors increases or as forecasting horizon lengthens, the forecast accuracy of the hard threshold model deteriorates more quickly than any other factor models; LARS and LARS(SPC) start to outperform the principal component models and the hard threshold models for all the density forecasting methods. Since the distribution of inflation is more volatile than that of output growth, the LARS(SPC) forecasts exhibit the best performance among all factors models for all density forecasting methods and forecasting horizons for inflation; while the best factor model for output growth forecast uses either LARS or LARS(SPC).

Moreover, Bayesian information criterion cannot select the appropriate *number of factors*. Given all else equal, density forecasts using BIC selected number of factors generally exhibit lower predictive likelihood than density forecasts using only a fixed number of factors. For example, as shown in Table $2.a$ and $2.b$, the predictive likelihood of factor-augmented models increases by 10 to 20 as the number of factors is reduced from what BIC selected to one. Point forecasting studies like Stock and Watson (2002b) and Faust and Wright (2009) have also found that the forecast accuracy is worse under BIC than under the choice of few fixed factors. Meanwhile, as the number of factors changes from a fixed number to that selected by BIC, factor-augmented density forecasts become less accurate than univariate autoregressive forecasts. Take forecasts of the quarterly output growth as an example: when the number of factors is set to be one, factor-augmented models exhibit higher predictive likelihood than models using only univariate autoregressive lags[5](see Table 2.d). When the number of factors is BIC-determined, the predictive likelihood of factor models falls below that of the autoregressive models at all forecasting horizons for all the density forecasting methods[6](see Table 2.c).

5 Conclusion

This paper finds that density forecasts using quantile regression is significantly more accurate than two conventional conditional-mean-centered density forecasts in forecasting the conditional distribution of future U.S. inflation and output growth. This paper also contributes to the empirical work on factor models, especially that of Stock and Watson (1998), by providing empirical evidence that including factors in the quantile density forecast can improve the forecast accuracy when our goal is to forecast the entire conditional distribution of future inflation and output growth. Certain linear factor selection rules such as Bayesian information criterion and hard threshold can not select factors that adapt to density forecasting well. The best way of selecting predictors for density forecasting is based upon Least Angle Regression. It would be interesting to develop model selection rules that can select factors especially powerful in density forecasting, based on the distributional connections between the variable to be forecasted and the predictors.

[5]The sole exception appears in quantile density forecast at $h = 1$.

[6]The sole exception is semi-parametric PC-model at $h = 1$.

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A Bootstrap Procedure

- *Step* 1: For each model k, write equation (4) in companion form yields:

$$\begin{bmatrix} Y_{t-1+h} \\ F_{t-1+h} \end{bmatrix} = \begin{bmatrix} \alpha(L) & \beta \\ 0 & \Phi(L) \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ F_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{Y_{t-1+h}} \\ G\varepsilon_{f_{t-1+h}} \end{bmatrix} \tag{10}$$

  Y_t is $M \times 1$. F_t is r. The constant term γ is included in the regression but not written here for notation simplicity. Since the true number of dynamic factors is q. G is set to be $r \times q$. ε_{Y_t} and ε_{f_t} are structural shocks. Write the above VAR into MA form :

$$A(L)X_{t-1+h} = v_{t-1+h} = R\varepsilon_{t-1+h} = \begin{bmatrix} I_M & 0_{M\times q} \\ 0_{r\times M} & G_{r\times q} \end{bmatrix} \begin{bmatrix} \varepsilon_{Y_{t-1+h}} \\ \varepsilon_{f_{t-1+h}} \end{bmatrix}, \tag{11}$$

  in which $E(\varepsilon_{t-1+h}\varepsilon'_{t-1+h}) = I$ and $E(v_{t-1+h}v'_{t-1+h}) = \Sigma = R\varepsilon_{t-1+h}\varepsilon'_{t-1+h}R' = RR'$.

- *Step* 2*:* To simulate normally distributed data series, the first step is to simulate Gaussian forecast errors while identifying R to preserve the inter-dependent relationship among innovations. The underlying structural shocks εs are generated from a multivariate normal distribution $N(0_{p\times 1}, I_p)$, where $p = M + r$. R is identified from Σ, the variance-covariance matrix of the residuals from the forecast using the last data vintage. For identification purpose, two restrictions are imposed here. The first restriction assumes innovations to Y_{t-1+h} have no contemporaneous effects on factors. The second restriction assumes innovations to factors that explain less variation in the many predictors have no contemporaneous effect on factors that explain more variation. The artificial forecast errors later used for bootstrap are constructed in way that

$$\hat{v}_{t-1+h} = \hat{R}\varepsilon_{t-1+h} = \begin{bmatrix} I_M & 0_{M\times 1} & 0_{M\times 1} & \dots & 0_{M\times 1} \\ 0_{1\times M} & \gamma^f_{11} & 0 & \dots & 0 \\ 0_{1\times M} & \gamma^f_{21} & \gamma^f_{22} & \dots & 0 \\ \vdots & & & & \vdots \\ 0_{1\times M} & \gamma^f_{r1} & \gamma^f_{r2} & \dots & \gamma^f_{rr} \end{bmatrix} \begin{bmatrix} \varepsilon_{Y_{t-1+h}} \\ \varepsilon_{f_{1t-1+h}} \\ \varepsilon_{f_{2t-1+h}} \\ \vdots \\ \varepsilon_{f_{rt-1+h}} \end{bmatrix}. \tag{12}$$

- *Step* 3*:* I then re-sample bootstrap forecast errors v^{boot}_{t-1+h} from the normally distributed artificial forecast errors $\hat{v}_{t-1+h}$. The bootstrap data generating process follows

$$\begin{bmatrix} Y^{boot}_{t-1+h} \\ F^{boot}_{t-1+h} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}(L) & \hat{\beta} \\ 0 & \hat{\Phi}(L) \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ F_{t-1} \end{bmatrix} + v^{boot}_{t-1+h}, \tag{13}$$

in which $\hat{\alpha}(L)$, $\hat{\beta}$, $\hat{\Phi}(L)$ are OLS estimates. Each simulated sample then participates in a real-time recursive out-of-sample density forecast and yields a *AG test stat*. Every *AG test stat* simulated in this way satisfies the null hypothesis of equal forecast accuracy, all of which form *bootstrap distribution* later used to determine the significance.

B Predictor Selection

B.1 Hard Threshold

The method of hard thresholding simply uses t-test to determine whether the i-th predictor in P_{t-1} is significant in predicting Y_{t-1+h} while controlling the autoregressive lags of Y_{t-1}. The selection rule is purely based upon simple linear relationship between the variable to be forecasted and the predictors. Specifically,

- Let W_{t-1} be the vector of a constant γ and p lags of Y_{t-1}. For each $i = 1, \ldots, N$, regress Y^h_{t-1+h} on W_{t-1} and each predictor $P_{i,t-1}$. In the meantime, let t_i be the $t-$statistic associated with $P_{i,t-1}$.

- Obtain a ranking of the marginal predictive power of $P_{i,t-1}$ by sorting $|t_1|, |t_2|, \ldots, |t_N|$ in descending order.

- Let k^*_α be the total number of series whose $|t_i|$ exceeds some threshold significance level α. Forecasts using sample data takes $\alpha = 1.65$. Forecasts in the Monte Carlo takes $\alpha = 0.29$ to prevent null.

- Then the selected predictors $P_{1,t-1}, \ldots, P_{k^*_\alpha,t-1}$ form a new large predictor matrix $P_{t(\alpha)}$. The static factors $\mathbf{F}^{hard}_{t-1}$ are estimated by taking the principal components of $P_{t(\alpha)}$.

- Estimate the optimal number of factors r using Bayesian information criteria without constant term and lags of Y_t as specified in Stock and Watson (2002b), and

$$\underbrace{F^{hard}_{t-1}}_{r\times 1} \subset \mathbf{F}^{hard}_{t-1}.$$

B.2 Least Angle Regression

LARS Algorithm Let $\hat{\mu}_k$ be the current OLS estimate of Y^h_{t-1+h} using the k selected predictors: $\hat{\mu}_k = P_{t-1}\hat{\beta}(k)$ (coefficients corresponds to unselected factors are set to be zero). Define $\hat{c}^k = P'_{t-1}(Y^h_{t-1+h} - \hat{\mu}_k)$ to be the vector of current correlations at the k-th step selection. $\hat{c}^k_j$ is proportional to the correlation between predictor $P_{j,t-1}$ and the k-th step residuals.

Formally, the LARS algorithm begins at $\hat{\mu}_0 = 0$. Let K be the set of indices corresponding to variables with the largest absolute correlations.

$$\hat{C} = \max_j |\hat{c}^k_j| \qquad K = \{j : |\hat{c}^k_j| = |\hat{C}|\}. \tag{14}$$

Let $s_j = sign(\hat{c}^k_j)$ and define the active predictor matrix corresponding to K as

$$P_K = (s_j P_{j,t-1})_{j \in K}. \tag{15}$$

Let $G_K = P'_K P_K$ and $A_K = (1'_K G_K^{-1} 1_K)^{-1/2}$, where 1_K is a vector of ones equaling the size of K. The algorithm proceeds by finding the unit vector making equal acute angles with each predictor in P_K. The equiangular vector u_K can be defined as

$$u_K = P_K \omega_K, \tag{16}$$

where $\omega_K = A_K G^{-1} 1_K$. Given the inner product vector $a_K = P' u_K$, LARS then updates $\hat{\mu}$ as

$$\hat{\mu}_{k+1} = \hat{\mu}_k + \hat{\gamma} u_K \tag{17}$$

where

$$\hat{\gamma} = \min_{j \in K}^{+} \left(\frac{\hat{C} - \hat{c}_j}{A_K - a_j}, \frac{\hat{C} + \hat{c}_j}{A_K + a_j} \right) \tag{18}$$

"$\min^{+}$" indicates the minimum is taken over only the positive components.

Algorithm Illustration

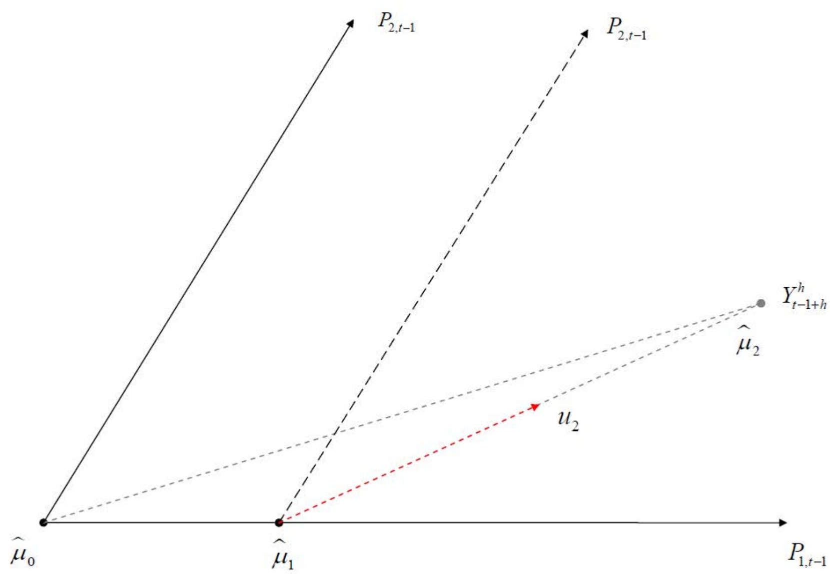


Figure B.2: The LARS algorithm in the case of $N = 2$ predictors; Y^{h}_{t-1+h} is the projection of Y^{h}_{t-1+h} into $\mathcal{L}(P_{1,t-1}, P_{2,t-1})$. Beginning at $\hat{\mu}_0 = 0$. The residual vector $Y^{h}_{t-1+h} - \hat{\mu}_0$ has greater correlation with predictor $P_{1,t-1}$ than with $P_{2,t-1}$. The algorithm then searches along the vector from $\hat{\mu}_0$ to $P_{1,t-1}$, and the next LARS estimate is $\hat{\mu}_1 = \hat{\mu}_0 + \hat{\gamma}_1 P_{1,t-1}$. $\hat{\gamma}_1$ is the step length of the search. In the $N = 2$ case, it is chosen such that $Y^{h}_{t-1+h} - \hat{\mu}_1$ bisects angle $\angle P_{2,t-1}\hat{\mu}_1 P_{1,t-1}$. Then, as the residual vector moves to be $Y^{h}_{t-1+h} - \hat{\mu}_1$, the second selection starts. After finding that the predictor 2 has the greatest current correlation, the search switches to another hyperplane spanned by $P_{2,t-1}$, $\hat{\mu}_2$, and Y^{h}_{t-1+h}. $\hat{\mu}_2 = \hat{\mu}_1 + \hat{\gamma}_2 u_2$, where u_2 is the unit bisector; $\hat{\mu}_2 = Y^{h}_{t-1+h}$ in the case $N = 2$, but not for the case $N > 2$.

Table 1.*a Real-time Density Forecast on* ***Monthly CPI (BIC)****: Sample Periods* 1959 : 01 − 2010 : 06

Forecast Horizon	$H=1M$			$H=3M$			$H=6M$			$H=12M$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
CPI-Hard Threshold	−1209.06	−1301.32	−1219.95	−1114.01	−1266.43	−1121.72	−1008.47	−1237.72	−1027.07	−1035.16	−1270.03	−1011.62
AG-stat		2.05	0.74		2.46	0.45		2.42	1.09		2.39	−1.00
Boot. p-value		(0.00*)	(0.07)		(0.00*)	(0.08)		(0.00*)	(0.01*)		(0.00*)	(0.86)
Asy. p-value		[0.02*]	[0.23]		[0.01*]	[0.33]		[0.01*]	[0.14]		[0.01*]	[0.84]
CPI-LARS(k)	−1213.36	−1286.31	−1215.01	−1110.17	−1248.03	1106.31	−1023.57	−1209.04	−1019.31	−1031.30	−1287.90	−1004.34
AG-stat		2.06	0.13		2.13	−0.27		2.04	−0.19		2.07	−1.07
Boot. p-value		(0.00*)	(0.33)		(0.00*)	(0.50)		(0.00*)	(0.45)		(0.00*)	(0.83)
Asy. p-value		[0.02*]	[0.45]		[0.02*]	[0.61]		[0.02*]	[0.58]		[0.02*]	[0.86]
CPI-LARS(SPC)	−1201.87	−1307.65	−1227.24	−1109.58	−1302.45	−1131.06	−1020.60	−1310.39	−1057.29	−1031.06	−1277.49	−1045.34
AG-stat		2.80	1.90		2.35	0.97		2.25	1.33		2.61	0.61
Boot. p-value		(0.00*)	(0.00*)		(0.00*)	(0.02*)		(0.03*)	(0.01*)		(0.00*)	(0.02*)
Asy. p-value		[0.00*]	[0.03*]		[0.01*]	[0.17]		[0.01*]	[0.09]		[0.00]	[0.27]
CPI-Principal Component	−1209.36	−1275.96	−1214.32	−1108.50	−1237.84	−1123.19	−1010.52	−1218.02	1042.20	−1033.18	−1209.27	−1028.36
AG-stat		1.88	0.37		1.98	0.84		2.19	1.48		2.12	−0.19
Boot. p-value		(0.00*)	(0.28)		(0.00*)	(0.02*)		(0.00*)	(0.00*)		(0.00*)	(0.17)
Asy. p-value		[0.03*]	[0.36]		[0.02*]	[0.20]		[0.01*]	[0.07]		[0.02*]	[0.57]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
CPI-QAR/AR	−1222.45	−1284.69	−1229.09	−1110.11	−1238.61	−1123.23	−1031.11	−1199.86	−1038.20	−1043.96	−1223.97	−1065.70
AG-stat		1.69	0.48		2.02	0.75		2.35	0.34		2.25	0.95
Boot. p-value		(0.00*)	(0.48)		(0.02*)	(0.27)		(0.04*)	(0.35)		(0.12)	(0.01*)
Asy. p-value		[0.05*]	[0.32]		[0.02*]	[0.23]		[0.01*]	[0.37]		[0.01*]	[0.17]

Note: $(\cdot)$ is the bootstrap p-values. $[\cdot]$ is the asymptotic p-values. $*$ indicates significance at 5% level. Epanechnikov kernel entails the choice of bandwidth $h_n = 2.34 \cdot \sigma_{\hat{Q}_{y_t}(\tau_i|X_{t-1})} \cdot n^{(-1/5)}$ $(n = 99)$, the rescaled versions of which are also considered: $0.8h_n$ is used in the monthly CPI autoregressive forecast; $0.5h_n$ is used in monthly GDP forecast. Define $\Delta\hat{Q}^{max}_{y^h_{t-1+h}}(\tau_i|X_{t-1})$ to be the maximum distance of all pairwise adjacent quantiles, I add two points that are $2 \cdot \Delta\hat{Q}^{max}_{y_t}(\tau_i|X_{t-1})$ away from the minimum and the maximum quantile estimates to smooth the tails.

Table 1.b Real-time Density Forecast on **Monthly CPI (one factor)**: *Sample Periods* 1959 : 01 − 2010 : 06

Forecast Horizon	$H = 1M$			$H = 3M$			$H = 6M$			$H = 12M$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
CPI-Hard Threshold	−1221.94	−1287.85	−1228.34	−1117.02	−1230.34	−1120.28	−1017.51	−1201.52	−1030.41	−1022.97	−1200.10	−1013.84
AG-stat		1.77	0.41		1.72	0.22		1.81	0.62		1.91	−0.47
Boot. p-value		(0.00*)	(0.13)		(0.01*)	(0.23)		(0.04*)	(0.06)		(0.06)	(0.39)
Asy. p-value		[0.04*]	[0.34]		[0.04*]	[0.41]		[0.04*]	[0.27]		[0.03*]	[0.68]
CPI-LARS(k)	−1228.36	−1290.33	−1234.06	−1123.62	−1251.32	1132.17	−1027.52	−1239.24	−1063.54	−1037.60	−1258.84	−1047.38
AG-stat		1.78	0.39		1.93	0.48		2.04	1.59		2.05	0.50
Boot. p-value		(0.00*)	(0.23)		(0.00*)	(0.10)		(0.01*)	(0.00*)		(0.03*)	(0.03*)
Asy. p-value		[0.04*]	[0.35]		[0.03*]	[0.32]		[0.02*]	[0.06]		[0.02*]	[0.31]
CPI-LARS(SPC)	−1236.91	−1284.82	−1227.94	−1122.11	−1237.85	−1123.67	−1039.63	−1200.54	−1037.29	−1050.40	−1226.63	−1067.47
AG-stat		1.28	−0.61		1.84	0.09		2.19	−0.12		2.08	0.80
Boot. p-value		(0.00*)	(0.62)		(0.00*)	(0.25)		(0.00*)	(0.40)		(0.03*)	(0.01*)
Asy. p-value		[0.10]	[0.73]		[0.03*]	[0.46]		[0.01*]	[0.55]		[0.02*]	[0.21]
CPI-Principal Component	−1224.64	−1287.84	−1233.12	−1122.33	−1247.16	−1136.80	−1029.01	−1220.12	1049.67	−1032.58	−1223.84	−1049.07
AG-stat		1.67	0.58		1.85	0.83		1.89	0.98		2.02	0.78
Boot. p-value		(0.00*)	(0.12)		(0.00*)	(0.03*)		(0.03*)	(0.03*)		(0.04*)	(0.03*)
Asy. p-value		[0.05*]	[0.28]		[0.03*]	[0.20]		[0.03*]	[0.16]		[0.02*]	[0.22]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
CPI-QAR/AR	−1222.45	−1284.69	−1229.09	−1110.11	−1238.61	−1123.23	−1031.11	−1199.86	−1038.20	−1043.96	−1223.97	−1065.70
AG-stat		1.69	0.48		2.02	0.75		2.35	0.34		2.25	0.95
Boot. p-value		(0.00*)	(0.48)		(0.02*)	(0.27)		(0.04*)	(0.35)		(0.12)	(0.01*)
Asy. p-value		[0.05*]	[0.32]		[0.02*]	[0.23]		[0.01*]	[0.37]		[0.01*]	[0.17]

Note: When the number of factors reduced from what BIC selected to one, the predictive likelihoods of quantile density forecast and semi-parametric density forecast both decline, while that of normal density forecast increases but still lower than that of the two. This indicates that some factors though may not have strong predictive power in the conditional mean, but may be crucial in forecasting the higher moments.

Table 1.c Real-time Density Forecast on ***Quarterly CPI (BIC)****: Sample Periods* 1959 : *Q*1 − 2010 : *Q*2

Forecast Horizon	$H = 1Q$			$H = 2Q$			$H = 4Q$			$H = 8Q$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
CPI-Hard Threshold	−343.46	−358.53	−370.33	−319.19	−361.01	−348.30	−327.30	−392.61	−325.48	−334.60	−577.66	−346.90
AG-stat		1.47	1.90		2.01	1.82		1.81	−0.20		1.52	1.24
Boot. p-value		(0.00*)	(0.04*)		(0.00*)	(0.03*)		(0.02*)	(0.80)		(0.06)	(0.08)
Asy. p-value		[0.07]	[0.03*]		[0.02*]	[0.03*]		[0.03*]	[0.58]		[0.06]	[0.11]
CPI-LARS(k)	−343.17	−370.50	−365.23	−318.44	−388.55	−343.27	−332.65	−442.62	−329.57	−379.84	−939.25	−372.13
AG-stat		2.08	1.75		2.61	1.87		1.64	−0.26		1.68	−0.25
Boot. p-value		(0.00*)	(0.05*)		(0.00*)	(0.07)		(0.02*)	(0.91)		(0.17)	(0.89)
Asy. p-value		[0.02*]	[0.04*]		[0.00*]	[0.03*]		[0.05*]	[0.60]		[0.05]	[0.60]
CPI-LARS(SPC)	−348.02	−458.48	−374.28	−331.63	−484.32	−358.44	−337.04	−534.06	−351.30	−364.74	−1179.26	−385.74
AG-stat		2.28	2.13		2.87	1.80		2.83	0.98		2.05	1.61
Boot. p-value		(0.00*)	(0.05*)		(0.00*)	(0.08)		(0.00*)	(0.18)		(0.08)	(0.01*)
Asy. p-value		[0.01*]	[0.02*]		[0.00*]	[0.04*]		[0.00*]	[0.16]		[0.02*]	[0.05*]
CPI-Principal Component	−342.05	−358.36	−364.36	−323.49	−361.41	−342.14	−326.17	−390.89	−327.71	−347.50	−569.26	−347.71
AG-stat		1.56	1.84		1.93	1.30		1.88	0.17		1.50	0.01
Boot. p-value		(0.00*)	(0.04*)		(0.00*)	(0.18)		(0.02*)	(0.64)		(0.35)	(0.60)
Asy. p-value		[0.06]	[0.03*]		[0.03*]	[0.10]		[0.03*]	[0.43]		[0.07]	[0.49]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
CPI-QAR/AR	−349.55	−366.38	−363.65	−318.68	−377.29	−347.59	−327.20	−421.19	−352.44	−339.48	−564.39	−354.42
AG-stat		1.60	1.65		2.21	1.87		2.05	2.18		1.62	1.65
Boot. p-value		(0.01*)	(0.15)		(0.00*)	(0.04*)		(0.00*)	(0.00*)		(0.23)	(0.01*)
Asy. p-value		[0.06]	[0.05*]		[0.01*]	[0.03*]		[0.02*]	[0.01*]		[0.05*]	[0.05*]

Note: When the comparison is between quantile density forecast and normal density forecast, significance exists despite various modeling specifications at all forecasting horizons. When the comparison is between quantile density forecast and semi-parametric density forecast, significance exists only at short horizons in factor models; at all horizons in quarterly autoregressive forecasts, but not at all in monthly autoregressive forecasts.

Table 1.d Real-time Density Forecast on ***Quarterly CPI (one factor)****: Sample Periods* $1959:Q1-2010:Q2$

Forecast Horizon	$H=1Q$			$H=2Q$			$H=4Q$			$H=8Q$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
CPI-Hard Threshold	−345.63	−366.82	−366.63	−318.92	−384.80	−351.63	−329.77	−447.44	−333.60	−373.10	−838.89	−377.40
AG-stat		1.59	1.82		2.24	2.10		2.01	0.39		2.07	0.17
Boot. p-value		(0.00*)	(0.06)		(0.00*)	(0.01*)		(0.00*)	(0.64)		(0.01*)	(0.66)
Asy. p-value		[0.06]	[0.03*]		[0.01*]	[0.02*]		[0.02*]	[0.35]		[0.02*]	[0.43]
CPI-LARS(k)	−346.69	−366.35	−369.39	−318.16	−362.23	−346.63	−326.50	−389.19	−323.26	−342.25	−575.34	−348.44
AG-stat		1.63	1.94		2.09	1.93		1.78	−0.38		1.49	0.53
Boot. p-value		(0.00*)	(0.02*)		(0.00*)	(0.02*)		(0.04*)	(0.84)		(0.40)	(0.27)
Asy. p-value		[0.05*]	[0.03*]		[0.02*]	[0.03*]		[0.04*]	[0.65]		[0.07]	[0.30]
CPI-LARS(SPC)	−351.77	−384.31	−379.57	−317.18	−373.49	−347.42	−329.94	−427.05	−345.92	−337.89	−560.25	−339.20
AG-stat		1.98	2.08		2.05	1.77		2.28	1.85		1.58	0.15
Boot. p-value		(0.00*)	(0.01*)		(0.00*)	(0.07)		(0.01*)	(0.02*)		(0.22)	(0.37)
Asy. p-value		[0.02*]	[0.02*]		[0.02*]	[0.04*]		[0.01*]	[0.03*]		[0.06]	[0.44]
CPI-Principal Component	−348.13	−364.87	−363.20	−323.91	−376.31	−343.11	−339.46	−472.66	−346.69	−372.53	−714.55	−375.63
AG-stat		1.38	1.31		2.11	1.34		1.58	0.59		1.80	0.14
Boot. p-value		(0.00*)	(0.24)		(0.00*)	(0.26)		(0.02*)	(0.61)		(0.07)	(0.66)
Asy. p-value		[0.08]	[0.09]		[0.02*]	[0.09]		[0.06]	[0.28]		[0.04*]	[0.44]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
CPI-QAR/AR	−349.55	−366.38	−363.65	−318.68	−377.29	−347.59	−327.20	−421.19	−352.44	−339.48	−564.39	−354.42
AG-stat		1.60	1.65		2.21	1.87		2.05	2.18		1.62	1.65
Boot. p-value		(0.01*)	(0.15)		(0.00*)	(0.04*)		(0.00*)	(0.00*)		(0.23)	(0.01*)
Asy. p-value		[0.06]	[0.05*]		[0.01*]	[0.03*]		[0.02*]	[0.01*]		[0.05*]	[0.05*]

Note: The general pattern in all three density forecast methods is that: as the number of factor reduced to one, the forecast accuracy of LARS-based models increases, while that of hard threshold model and simple PC model decreases. This indicates that in density forecast the Least Angle Regression is an effective method in selecting the most relevant predictors.

Table 2.a Real-time Density Forecast on ***Monthly GDP (BIC)****: Sample Periods* 1959 : 01 − 2010 : 06

Forecast Horizon	$H = 1M$			$H = 3M$			$H = 6M$			$H = 12M$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
GDP-Hard Threshold	−872.65	−866.71	−893.44	−633.26	−630.79	−627.16	−622.49	−639.91	−640.27	−671.89	−652.30	−637.67
AG-stat		−0.67	1.35		−0.25	−0.34		0.79	0.45		−0.97	−0.67
Boot. p-value		(0.06)	0.19		(0.10)	(0.59)		(0.06)	(0.02*)		(0.68)	(0.05*)
Asy. p-value		[0.75]	[0.09]		[0.60]	[0.63]		[0.21]	[0.33]		[0.83]	[0.75]
GDP-LARS(k)	−883.76	−877.58	−903.84	−624.71	−646.31	−648.23	−622.78	−620.61	−609.89	−692.01	−676.93	−631.02
AG-stat		−0.52	1.39		1.26	1.23		−0.13	−0.37		−0.56	−1.15
Boot. p-value		(0.03*)	(0.17)		(0.00*)	(0.05*)		(0.26)	(0.07)		(0.60)	(0.15)
Asy. p-value		[0.70]	[0.08]		[0.10]	[0.11]		[0.55]	[0.64]		[0.71]	[0.87]
GDP-LARS(SPC)	−863.04	−873.18	−894.85	−632.21	−714.16	−679.61	−643.60	−753.62	−681.27	−725.93	−835.39	−648.31
AG-stat		0.99	2.29		2.56	−2.73		−2.51	−1.42		−2.00	−1.54
Boot. p-value		(0.00*)	(0.00*)		(0.00*)	(0.00*)		(0.00*)	(0.00*)		(0.00*)	(0.35)
Asy. p-value		[0.16]	[0.01*]		[0.01*]	[0.00*]		[0.01*]	[0.08]		[0.02*]	[0.94]
GDP-Principal Component	−869.19	−865.18	−874.95	−637.49	−634.65	−625.62	−642.68	−629.81	−619.60	−655.87	−639.24	−624.65
AG-stat		−0.45	0.64		−0.25	−0.72		−0.89	−0.69		−0.82	−0.80
Boot. p-value		(0.04*)	0.55		(0.04*)	(0.72)		(0.27)	(0.26)		(0.59)	(0.17)
Asy. p-value		[0.67]	[0.26]		[0.60]	[0.76]		[0.81]	[0.76]		[0.79]	[0.79]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
GDP-QAR/AR	−845.21	−851.15	−860.96	−647.95	−670.13	−656.95	−626.59	−651.32	−631.87	−628.50	−629.90	−584.40
AG-stat		1.08	1.08		2.71	0.70		1.54	0.23		0.08	−1.12
Boot. p-value		(0.00*)	0.60		(0.00*)	(0.37)		(0.00*)	(0.29)		(0.12)	(0.55)
Asy. p-value		[0.14]	[0.14]		[0.00*]	[0.24]		[0.06]	[0.41]		[0.47]	[0.87]

Note: (·) is the bootstrap p-values. [·] is the asymptotic p-values. ∗ indicates significance at 5% level. Epanechnikov kernel is chosen to estimate predictive likelihoods. The bandwidth are chosen to be $0.5h_n$. The quantile-varying bandwidth choices suggested by Hall and Sheather (1988) and Bofingeb (1975) are also considered. In practice, the quantile density estimation even with a constant bandwidth choice still outperforms the mean-centered density forecast methods.

Table 2.b Real-time Density Forecast on ***Monthly GDP (one factor)****: Sample Periods* 1959 : 01 − 2010 : 06

Forecast Horizon	$H = 1M$			$H = 3M$			$H = 6M$			$H = 12M$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
GDP-Hard Threshold	−862.12	−866.26	−902.66	−617.99	−628.31	−624.55	−616.20	−622.30	−610.86	−637.35	−656.09	−577.28
AG-stat		0.53	2.22		1.31	0.50		0.50	−0.21		0.74	−1.73
Boot. p-value		(0.00*)	(0.01*)		(0.00*)	(0.35)		(0.02*)	(0.23)		(0.04*)	(0.77)
Asy. p-value		[0.30]	[0.01*]		[0.09]	[0.31]		[0.31]	[0.58]		[0.23]	[0.96]
GDP-LARS(k)	−852.41	−858.28	−886.98	−634.16	−639.05	−622.83	−636.41	−653.36	−613.90	−635.81	−653.37	−587.38
AG-stat		0.72	2.24		0.69	−0.75		1.07	−0.72		0.54	−1.20
Boot. p-value		(0.00*)	(0.02*)		(0.01*)	(0.80)		(0.01*)	(0.50)		(0.05*)	0.40
Asy. p-value		[0.24]	[0.01*]		[0.24]	[0.77]		[0.14]	[0.76]		[0.29]	[0.88]
GDP-LARS(SPC)	−846.41	−851.46	−859.28	−645.40	−670.26	−655.99	−631.50	−663.09	−640.07	−656.14	−671.32	−602.36
AG-stat		0.82	1.07		2.31	0.84		1.80	0.36		0.54	−1.26
Boot. p-value		(0.00*)	(0.38)		(0.00*)	(0.17)		(0.00*)	(0.14)		(0.06)	(0.51)
Asy. p-value		[0.21]	[0.14]		[0.01*]	[0.20]		[0.04*]	[0.36]		[0.30]	[0.90]
GDP-Principal Component	−848.30	−854.28	−876.63	−634.00	−644.42	−632.67	−618.76	−636.88	−627.14	−638.35	−638.96	−578.88
AG-stat		0.87	2.14		1.26	−0.09		1.06	0.32		0.03*	−1.50
Boot. p-value		(0.00*)	0.04*		(0.00*)	(0.60)		(0.01*)	(0.13)		(0.14)	(0.56)
Asy. p-value		[0.19]	[0.02*]		[0.10]	[0.54]		[0.15]	[0.37]		[0.49]	[0.93]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
GDP-QAR/AR	−845.21	−851.15	−860.96	−647.95	−670.13	−656.95	−626.59	−651.32	−631.87	−628.50	−629.90	−584.40
AG-stat		1.08	1.08		2.71	0.70		1.54	0.23		0.08	−1.12
Boot. p-value		(0.00*)	0.60		(0.00*)	(0.37)		(0.00*)	(0.29)		(0.12)	(0.55)
Asy. p-value		[0.14]	[0.14]		[0.00*]	[0.24]		[0.06]	[0.41]		[0.47]	[0.87]

Note: Quantile density forecast significantly outperforms the normal density forecast at all horizons but significantly outperforms the semi-parametric density forecast only at short forecasting horizon. As the number of factors reduced from what BIC selected to one, the forecast accuracy of all three methods increases. This indicates that BIC does not work well in density forecasting of future output growth.

Table 2.c Real-time Density Forecast on ***Quarterly GDP (BIC)****: Sample Periods* 1959 : *Q*1 − 2010 : *Q*2

Forecast Horizon	$H = 1Q$			$H = 2Q$			$H = 4Q$			$H = 8Q$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
GDP-Hard Threshold	−419.71	−444.40	−422.94	−382.00	−419.47	−409.39	−450.64	−477.14	−416.03	−467.30	−690.32	−371.24
AG-stat		1.09	0.24		1.77	1.93		0.84	−1.59		1.69	−2.44
Boot. p-value		(0.01*)	(0.24)		(0.00*)	(0.01*)		(0.10)	(0.81)		(0.01*)	(0.99)
Asy. p-value		[0.14]	[0.40]		[0.04*]	[0.03*]		[0.20]	[0.94]		[0.05*]	[0.99]
GDP-LARS(k)	−419.16	−438.60	−412.61	−402.10	−418.67	−408.55	−441.03	−485.17	−405.86	−432.82	−698.27	−368.25
AG-stat		1.29	−0.69		0.76	0.41		1.11	−1.65		1.54	−2.21
Boot. p-value		(0.00*)	(0.74)		(0.01*)	(0.47)		(0.04*)	(0.77)		(0.03*)	(0.81)
Asy. p-value		[0.10]	[0.76]		[0.22]	[0.34]		[0.13]	[0.95]		[0.06]	[0.99]
GDP-LARS(SPC)	−403.68	−485.10	−424.52	−405.68	−440.76	−400.84	−404.89	−550.64	−383.96	−398.61	−636.66	−349.73
AG-stat		1.74	1.59		1.04	−0.31		2.17	0.82		2.21	−1.54
Boot. p-value		(0.00*)	(0.00*)		(0.24)	(0.35)		(0.00*)	(0.10)		(0.00*)	(0.16)
Asy. p-value		[0.04*]	[0.06]		[0.15]	[0.62]		[0.01*]	[0.79]		[0.01*]	[0.94]
GDP-Principal Component	−405.64	−413.47	−399.50	−380.53	−421.93	−404.72	−396.65	−455.50	−405.93	−394.99	−482.94	−362.77
AG-stat		0.76	−0.65		1.66	2.04		1.38	0.53		1.95	−1.43
Boot. p-value		(0.00*)	(0.95)		(0.00*)	(0.01*)		(0.01*)	(0.21)		(0.01*)	(0.73)
Asy. p-value		[0.22]	[0.74]		[0.05*]	[0.02*]		[0.08]	[0.30]		[0.03*]	[0.92]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
GDP-QAR/AR	−400.37	−410.05	−411.52	−378.61	−392.46	−396.77	−375.50	−383.75	−382.89	−355.16	−370.68	−348.01
AG-stat		1.79	1.20		1.21	1.54		1.02	0.52		1.09	−0.49
Boot. p-value		(0.00*)	(0.40)		(0.00*)	(0.09)		(0.01*)	(0.33)		(0.09)	(0.42)
Asy. p-value		[0.04*]	[0.11]		[0.11]	[0.06]		[0.15]	[0.30]		[0.14]	[0.69]

Note: Quantile density forecast significantly outperforms the normal density forecast at all horizons but significantly outperforms the semi-parametric density forecast only at short forecasting horizon. When the number of factors is selected by BIC, forecasts conditional on autoregressive lags outperforms all factors models at all forecasting horizons under all density specifications. BIC does not work well in forecasting the conditional distribution of future output growth.

Table 2.d Real-time Density Forecast on ***Quarterly GDP (one factor)****: Sample Periods* $1959:Q1-2010:Q2$

Forecast Horizon	$H=1Q$			$H=2Q$			$H=4Q$			$H=8Q$		
with factors	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*	*Quantile FAVAR*	*Normal FAVAR*	*Semi.P. FAVAR*
GDP-Hard Threshold	−400.74	−403.52	−406.28	−370.27	−383.23	−391.74	−364.03	−369.85	−375.86	−361.88	−375.06	−334.82
AG-stat		0.45	0.62		1.89	1.91		0.59	0.83		0.83	1.36
Boot. p-value		(0.01*)	(0.61)		(0.00*)	(0.04*)		(0.05*)	(0.14)		(0.20)	(0.70)
Asy. p-value		[0.33]	[0.12]		[0.03*]	[0.07]		[0.28]	[0.20]		[0.20]	[0.91]
GDP-LARS(k)	−404.62	−408.47	−407.75	−382.36	−399.45	−397.21	−366.25	−370.54	−371.70	−355.50	−362.71	−323.00
AG-stat		0.47	0.30		1.10	1.48		0.40	0.41		0.48	−1.74
Boot. p-value		(0.00*)	(0.75)		(0.00*)	(0.06)		(0.03*)	(0.27)		(0.26)	(0.79)
Asy. p-value		[0.32]	[0.38]		[0.14]	[0.07]		[0.35]	[0.34]		[0.32]	[0.96]
GDP-LARS(SPC)	−400.83	−412.47	−408.62	−380.52	−432.09	−397.68	−382.52	−393.60	−387.83	−353.93	−387.47	−350.36
AG-stat		1.54	0.84		1.09	1.38		0.96	0.36		1.44	−0.25
Boot. p-value		(0.00*)	(0.00*)		(0.00*)	(0.00*)		(0.00*)	(0.00*)		(0.00*)	(1.00)
Asy. p-value		[0.06]	[0.20]		[0.14]	[0.08]		[0.17]	[0.36]		[0.07]	[0.60]
GDP-Principal Component	−404.67	−405.87	−402.19	−379.00	−389.42	−399.80	−366.69	−371.35	−371.46	−345.11	−364.06	−327.56
AG-stat		0.20	−0.26		1.15	1.83		0.49	0.39		1.28	−1.20
Boot. p-value		(0.02*)	(0.91)		(0.00*)	(0.02*)		(0.04*)	(0.28)		(0.07)	(0.62)
Asy. p-value		[0.42]	[0.60]		[0.13]	[0.03*]		[0.31]	[0.35]		[0.10]	[0.89]
without factors	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*	*QAR*	*AR*	*s.p.AR*
GDP-QAR/AR	−400.37	−410.05	−411.52	−378.61	−392.46	−396.77	−375.50	−383.75	−382.89	−355.16	−370.68	−348.01
AG-stat		1.79	1.20		1.21	1.54		1.02	0.52		1.09	−0.49
Boot. p-value		(0.00*)	(0.40)		(0.00*)	(0.09)		(0.01*)	(0.33)		(0.09)	(0.42)
Asy. p-value		[0.04*]	[0.11]		[0.11]	[0.06]		[0.15]	[0.30]		[0.14]	[0.69]

Note: When the number of factors is reduced to one, the predictive likelihood of all factor models largely increases. Factor models provide more accurate density forecast than the univariate autoregression forecasts at all forecasting horizons under various density specification. This indicates including only the first few factors might improve the forecast accuracy upon the univariate forecasts.

Table 3.a Kolmogorov Smirnov Test for ***Quarterly*** *Data*

(Sample Periods 1959 : Q1 − 2010 : Q2)

Forecast Horizon	$H = 1Q$		$H = 2Q$		$H = 4Q$		$H = 8Q$	
with factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
CPI-Hard Threshold	0.15	0.13	0.12	0.34	0.13	0.24	0.11	0.37
CPI-LARS(k)	0.14	0.14	0.14	0.17	0.13	0.25	0.22	(0.00*)
CPI-LARS(SPC)	0.16	0.09	0.14	0.17	0.15	0.12	0.19	(0.02*)
CPI-Principal Component	0.13	0.26	0.13	0.22	0.11	0.46	0.13	0.20
without factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
CPI-QAR/AR	0.09	0.70	0.09	0.62	0.12	0.28	0.14	0.14
with factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
GDP-Hard Threshold	0.11	0.42	0.09	0.70	0.14	0.14	0.19	(0.02*)
GDP-LARS(k)	0.10	0.53	0.15	0.13	0.15	0.10	0.19	(0.02*)
GDP-LARS(SPC)	0.07	0.90	0.15	0.13	0.20	(0.02*)	0.22	(0.00*)
GDP-Principal Component	0.07	0.92	0.07	0.87	0.10	0.51	0.15	0.13
without factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
GDP-QAR/AR	0.11	0.38	0.13	0.22	0.17	0.06	0.23	(0.00*)

Table 3.b Kolmogorov Smirnov Test for ***Monthly*** *Data*

(Sample Periods 1959 : 01 − 2010 : 06)

Forecast Horizon	$H = 1M$		$H = 3M$		$H = 6M$		$H = 12M$	
with factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
CPI-Hard Threshold	0.09	0.56	0.12	0.19	0.09	0.49	0.11	0.25
CPI-LARS(k)	0.11	0.27	0.12	0.20	0.11	0.30	0.13	0.11
CPI-LARS(SPC)	0.03	0.99	0.03	0.99	0.04	0.99	0.07	0.98
CPI-Principal Component	0.08	0.68	0.12	0.17	0.10	0.39	0.11	0.22
without factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
CPI-QAR/AR	0.05	0.98	0.07	0.76	0.05	0.98	0.05	0.97
with factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
GDP-Hard Threshold	0.06	0.94	0.07	0.74	0.08	0.62	0.12	0.20
GDP-LARS(k)	0.04	1.00	0.04	1.00	0.07	0.86	0.11	0.23
GDP-LARS(SPC)	0.08	0.64	0.10	0.31	0.15	(0.03*)	0.20	(0.00*)
GDP-Principal Component	0.04	1.00	0.06	0.90	0.08	0.58	0.09	0.47
without factors	*KS.stat.*	*p.val.*	*KS.stat.*	*p.val.*	*K.S.stat.*	*p.val.*	*K.S.stat*	*p.val.*
GDP-QAR/AR	0.06	0.87	0.11	0.27	0.14	0.08	0.16	(0.02*)

Note: (∗) indicates significance at 5% level. Acceptance indicates a "good fit" of the quantile density forecast.

Table 4.a **Monte Carlo Size I: DGP- Gaussian AR** $y_t = \tilde{\alpha} y_{t-1} + P \cdot u_{y,t} \sim N(0, \frac{P^2}{1-\tilde{\alpha}^2})$

Sample size & Num.of Replication	$T = 210,$	$Rep = 500$			$T = 622,$	$Rep = 500$		
Forecast Horizon	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
Quantile-AR 95%*int. Coverage Ratio*	0.87	0.87	0.87	0.84	0.90	0.89	0.89	0.88
Gaussian AR 95%*int. Coverage Ratio*	0.94	0.93	0.92	0.89	0.95	0.94	0.93	0.92
Semi.Param. AR 95%*int. Coverage Ratio*	0.87	0.86	0.85	0.83	0.95	0.94	0.93	0.92
Quantile-AR 99%*int. Coverage Ratio*	0.97	0.98	0.99	1.00	0.98	0.98	0.97	0.97
Quantile-AR vs. Gaussian AR	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
5% *Rej.Freq.*	0.00	0.00	0.00	0.04	0.00	0.00	0.00	0.00
10% *Rej.Freq.*	0.00	0.00	0.01	0.11	0.00	0.00	0.00	0.02
Quantile-AR vs. Semi.Param. AR	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
5% *Rej.Freq.*	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10% *Rej.Freq.*	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
5% **Rej.Freq. KS Test**	0.00	0.01	0.00	−−	0.00	0.00	0.02	0.02

Table 4.b **Monte Carlo Size II: DGP - FAVAR (Gaussian)** $y_t = \tilde{\alpha} y_{t-1} + \tilde{\beta} f_{t-1} + P \cdot u_{y,t}$

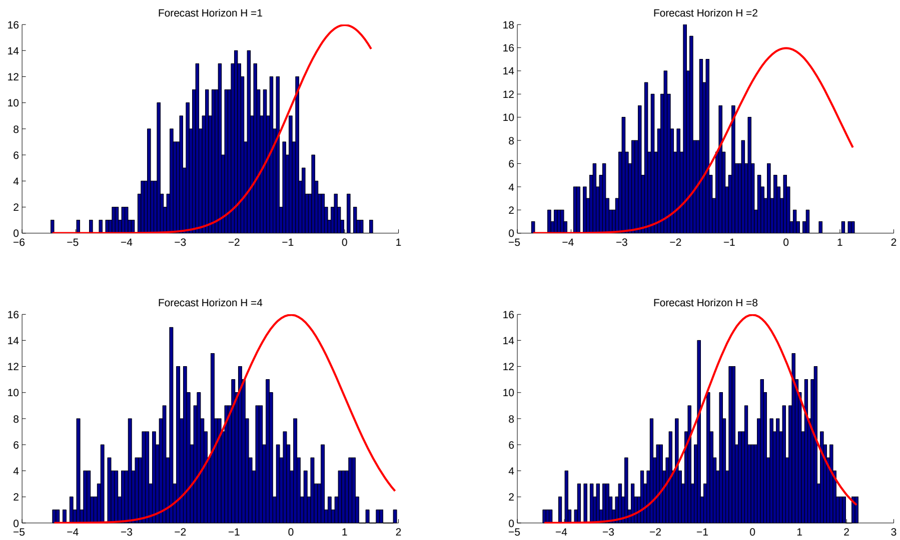
Sample size & Num.of Replication	$T = 210,$	$Rep = 500$			$T = 622,$	$Rep = 500$		
Forecast Horizon	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
Quantile-FAVAR 95%*int. Coverage Ratio*	0.87	0.87	0.87	0.84	0.89	0.89	0.88	0.88
FAVAR (Gaussian) 95%*int. Coverage Ratio*	0.94	0.93	0.92	0.89	0.95	0.94	0.93	0.92
Semi.Param. FAVAR 95%*int. Coverage Ratio*	0.87	0.86	0.85	0.83	0.89	0.88	0.88	0.87
Quantile-FAVAR 99%*int. Coverage Ratio*	0.97	0.98	0.99	1.00	0.98	0.97	0.97	0.96
Quantile-FAVAR vs. FAVAR (Gaussian)	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
5% *Rej.Freq.*	0.00	0.00	0.00	0.04	0.00	0.00	0.00	0.00
10% *Rej.Freq.*	0.00	0.00	0.01	0.13	0.00	0.00	0.00	0.01
Quantile-FAVAR vs. Semi.Param. FAVAR	$H = 1Q$	$H = 2Q$	$H = 4Q$	$H = 8Q$	$H = 1M$	$H = 3M$	$H = 6M$	$H = 12M$
5% *Rej.Freq.*	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10% *Rej.Freq.*	0.12	0.05	0.00	0.00	0.00	0.00	0.00	0.01
5% **Rej.Freq. KS Test**	0.00	0.01	0.00	−−	0.00	0.00	0.01	0.00

Notes: *Table 4.a,b* shows when density forecast models are nested, *AG test* in finite-sample is generally undersized, in which case the asymptotic critical value might signal that the quantile density forecast is not superior in cases where the quantile density forecast is superior.

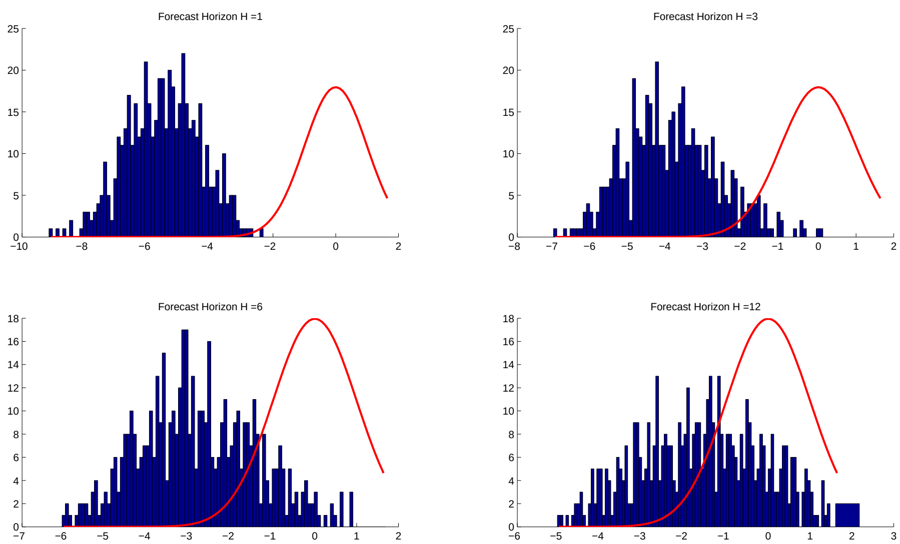
– Due to the overlapping structure in real-time forecast, I discard the overlapping part to ensure an uncorrelated white noise Z_t as suggested by Weiss (1978). But quarterly sample is too short, there is no way to perform this modification for $h = 8$.

Figure.1.a Quantile-AR vs. Gaussian AR (Size)

$T = 210$



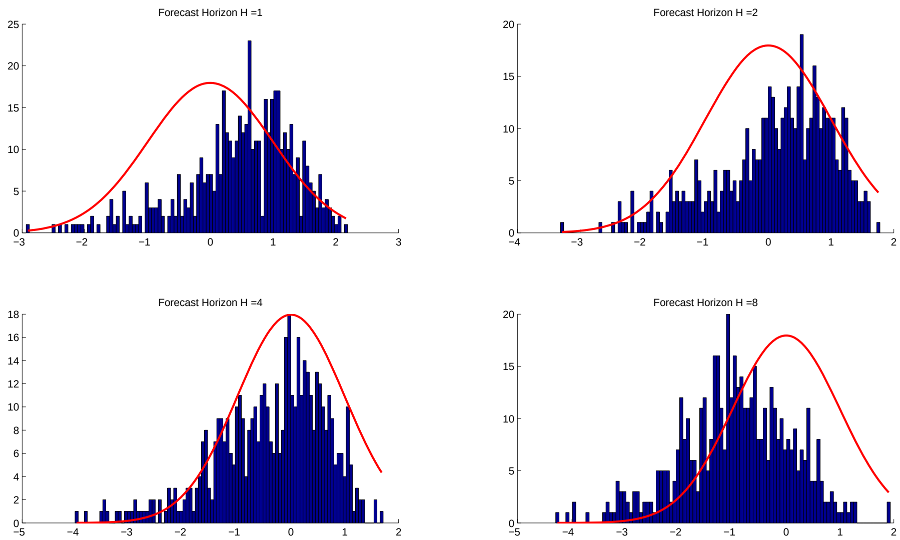
$T = 622$



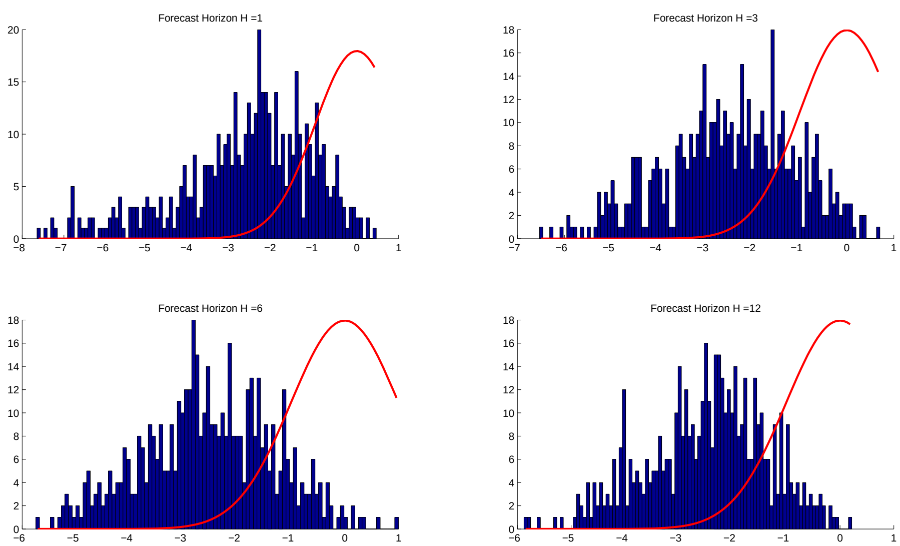
Notes: (Without factors) red line is the standard normal distribution. Histogram is the asymptotic distribution of $AG - stat$ under the null. As sample size increases, the asymptotic distribution of the AG *stat* is skewed to the right with respect to the asymptotic one. The comparison is between quantile density forecast and normal density forecast.

Figure.1.b Quantile-AR vs. Semi.Param-AR (Size)

$T = 210$



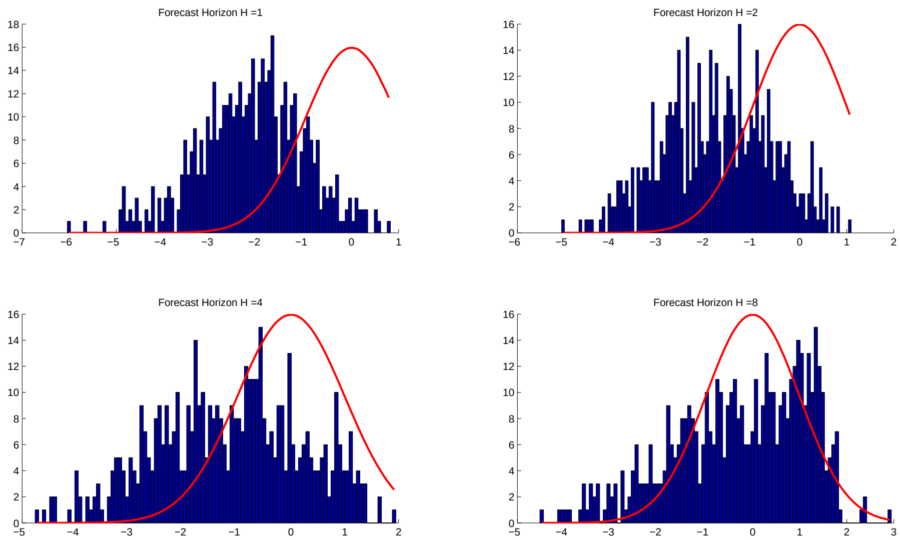
$T = 622$



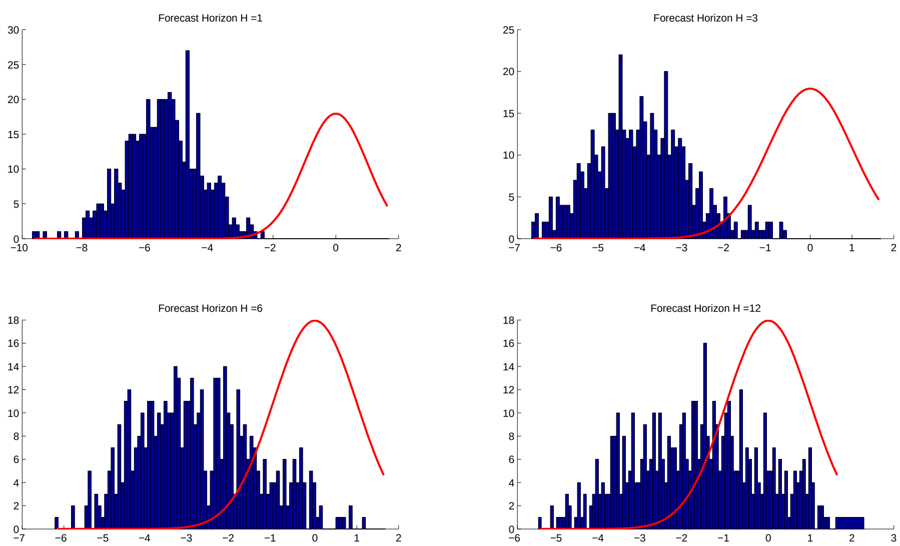
Notes: (Without factors) red line is the standard normal distribution. Histogram is the asymptotic distribution of $AG-stat$ under the null. As sample size increases, the asymptotic distribution of the AG $stat$ is skewed to the right with respect to the asymptotic one. The comparison is between quantile density forecast and semi-parametric density forecast.

Figure.2.a Quantile-FAVAR vs. FAVAR (Gaussian) (Size)

T = *210*



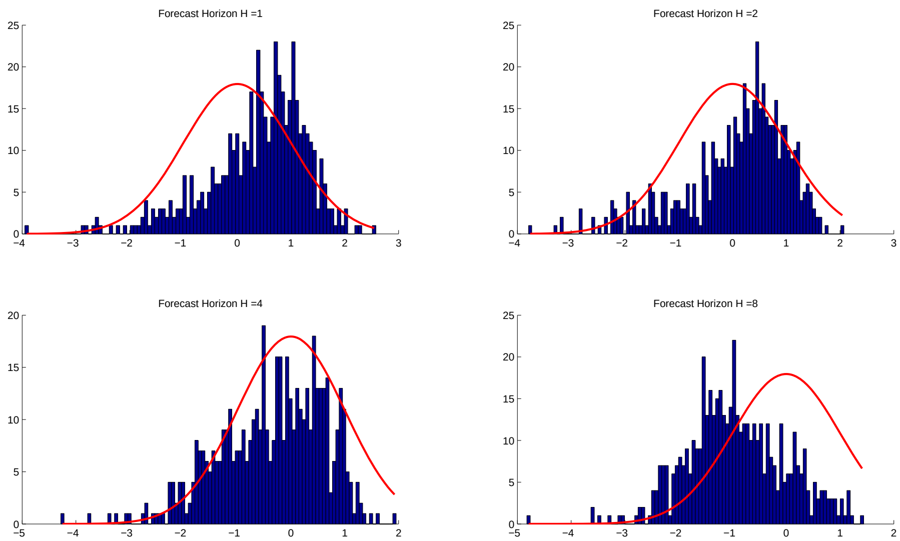
T = *622*



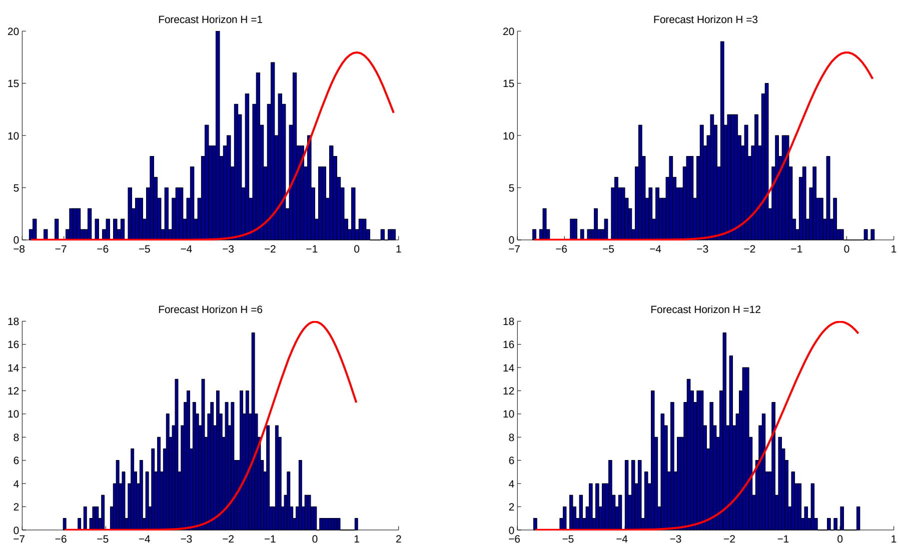
Notes: (In presence of factors) red line is the standard normal distribution. Histogram is the asymptotic distribution of $AG - stat$ under the null. As sample size increases, the asymptotic distribution of the AG $stat$ is skewed to the right with respect to the asymptotic one. The comparison is between quantile density forecast and normal density forecast.

Figure.2.b Quantile-FAVAR vs. Semi.Param. FAVAR (Size)

$T = 210$



$T = 622$



Notes: (In presence of factors) red line is the standard normal distribution. Histogram is the asymptotic distribution of $AG - stat$ under the null. As sample size increases, the asymptotic distribution of the $AG\ stat$ is skewed to the right with respect to the asymptotic one. The comparison is between quantile density forecast and semi-parametric density forecast.

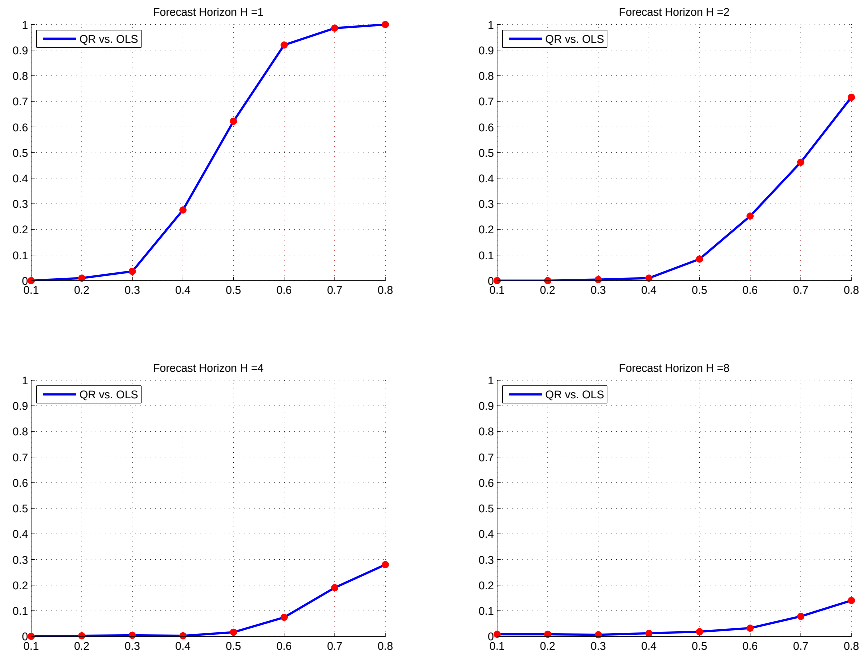
Table 5 **Monte Carlo Power: DGP** $y_t = \tilde{\alpha} + \tilde{\beta}(U_t)y_{t-1} + \Phi^{-1}(U_t)$, $U_t \sim [0\ 1]$.

Sample size & Num.of Replication	*T=210, Rep= 500*				*T=622, Rep= 500*			
Methods	**QAR vs. Gaussian AR**							
Forecast Horizon	H=1Q	H=2Q	H=4Q	H =8Q	H=1M	H=3M	H=6M	H =12M
c = 0.1	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
c = 0.2	0.01	0.00	0.00	0.01	0.03	0.00	0.00	0.01
c = 0.3	0.04	0.00	0.00	0.01	0.37	0.00	0.00	0.00
c = 0.4	0.28	0.01	0.00	0.01	0.91	0.03	0.01	0.01
c = 0.5	0.62	0.08	0.02	0.02	1.00	0.14	0.03	0.01
c = 0.6	0.92	0.25	0.07	0.03	1.00	0.53	0.10	0.06
c = 0.7	0.99	0.46	0.19	0.08	1.00	0.89	0.37	0.10
c = 0.8	1.00	0.72	0.28	0.14	1.00	0.97	0.61	0.23
Methods	**QAR vs. Semi.Param AR**							
Forecast Horizon	H=1Q	H=2Q	H=4Q	H =8Q	H=1M	H=3M	H=6M	H =12M
c = 0.1	0.10	0.02	0.01	0.00	0.07	0.00	0.00	0.00
c = 0.2	0.14	0.03	0.00	0.00	0.30	0.00	0.00	0.00
c = 0.3	0.34	0.04	0.01	0.00	0.73	0.01	0.00	0.00
c = 0.4	0.64	0.08	0.01	0.00	0.97	0.03	0.00	0.00
c = 0.5	0.85	0.22	0.02	0.00	1.00	0.09	0.00	0.00
c = 0.6	0.97	0.39	0.04	0.00	1.00	0.25	0.01	0.00
c = 0.7	0.99	0.56	0.11	0.01	1.00	0.55	0.02	0.00
c = 0.8	1.00	0.75	0.14	0.01	1.00	0.76	0.07	0.00

Notes: Except for long horizon, the power of the test increases as distance between c and 0 increases. The *AG test* exhibits under-power property either when c and 0 are too close or as the forecasting horizon lengthens. This indicates even when the quantile density forecast is superior than the mean-centered methods under some cases, the inference using asymptotic critical value may not suggest so.

Figure.3.a Quantile-AR vs. Gaussian AR (Power)

T = 210



T = 622

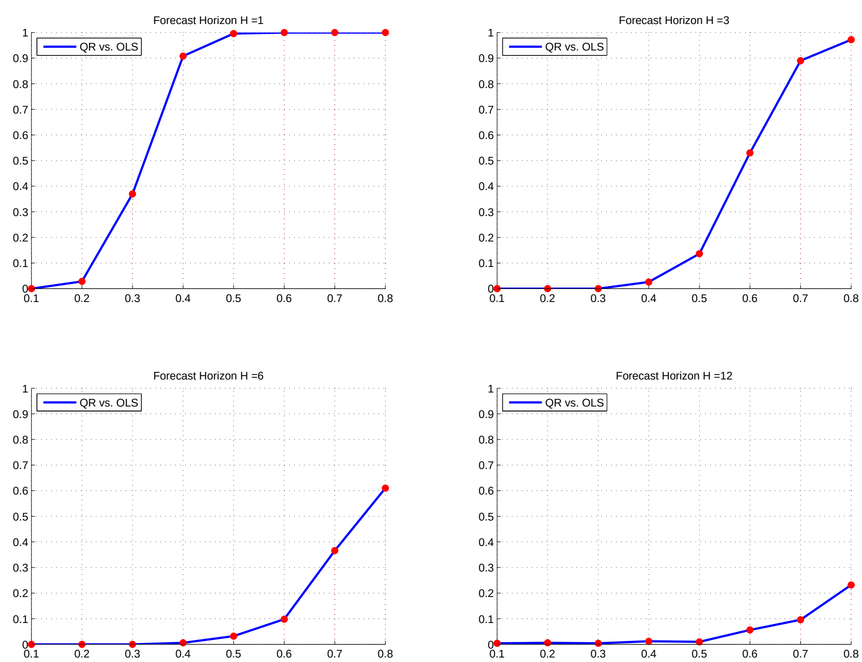
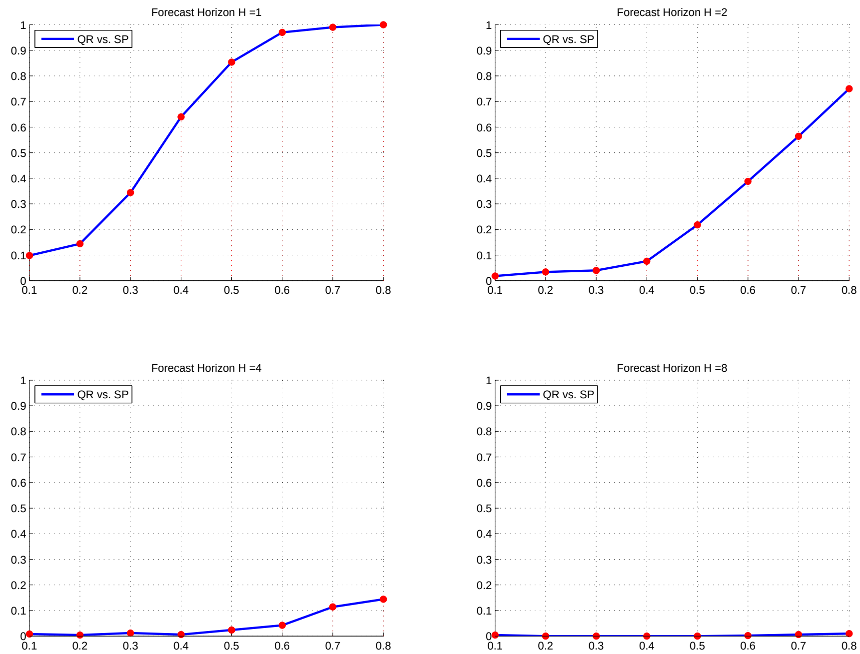


Figure.3.b Quantile-AR vs. Semi.Param AR (Power)

T = 210



T = 622

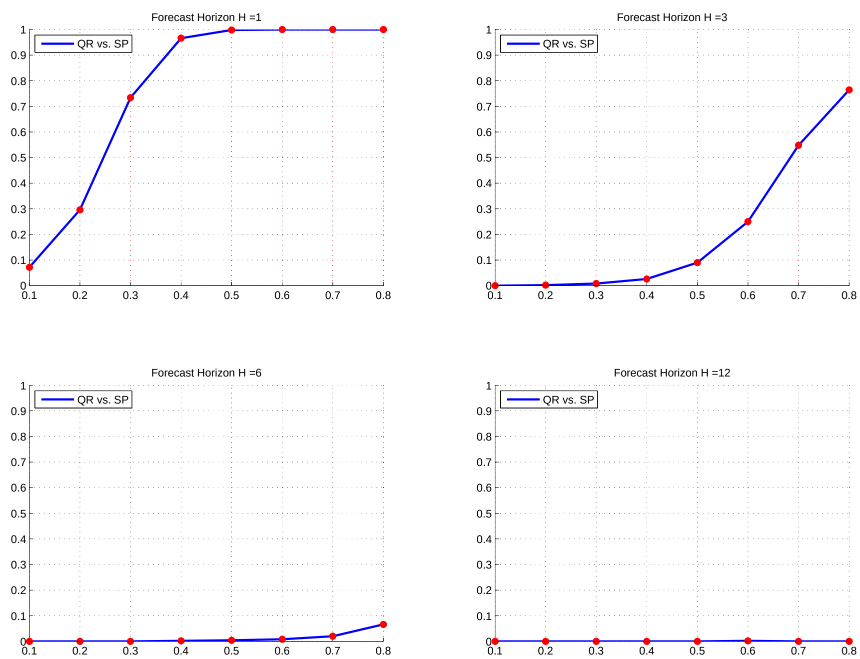


Figure.4.1.a. Gaussian AR versus FAVAR: Quarterly GDP

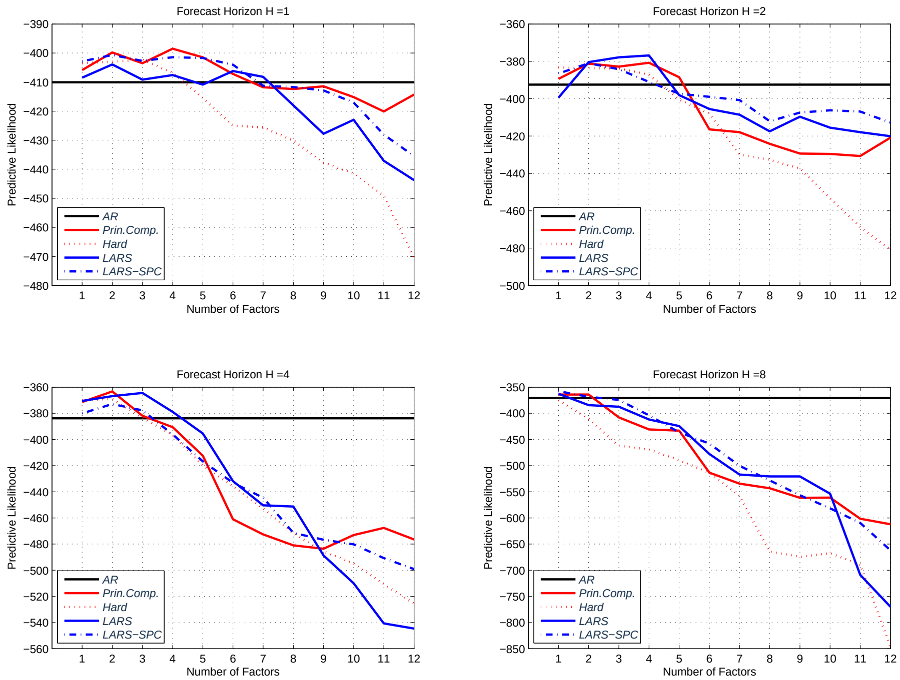


Figure.4.1.b. Gaussian AR versus FAVAR: Monthly CPI

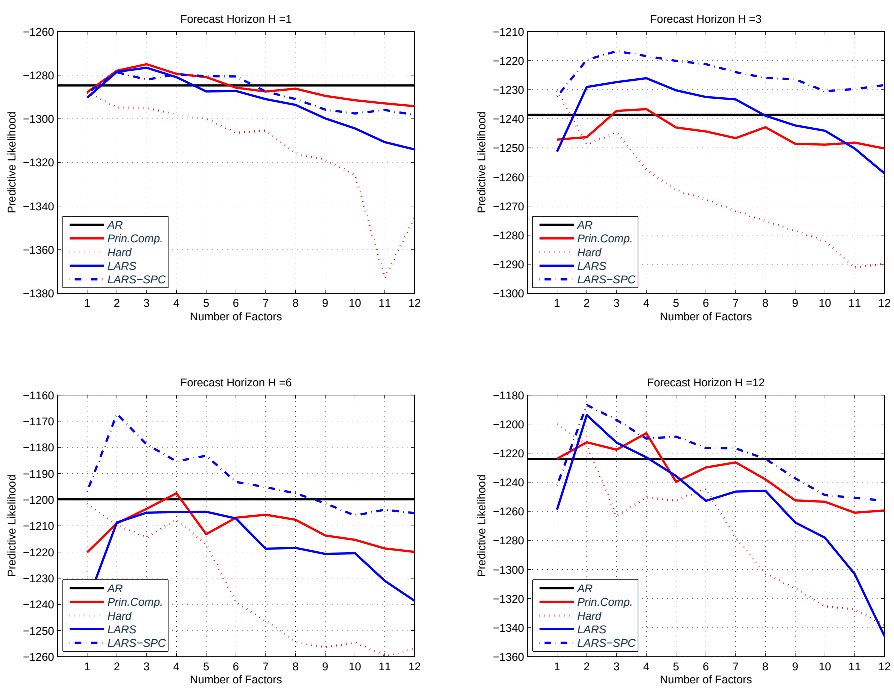


Figure.4.2.a QAR versus Quantile-FAVAR: Quarterly GDP

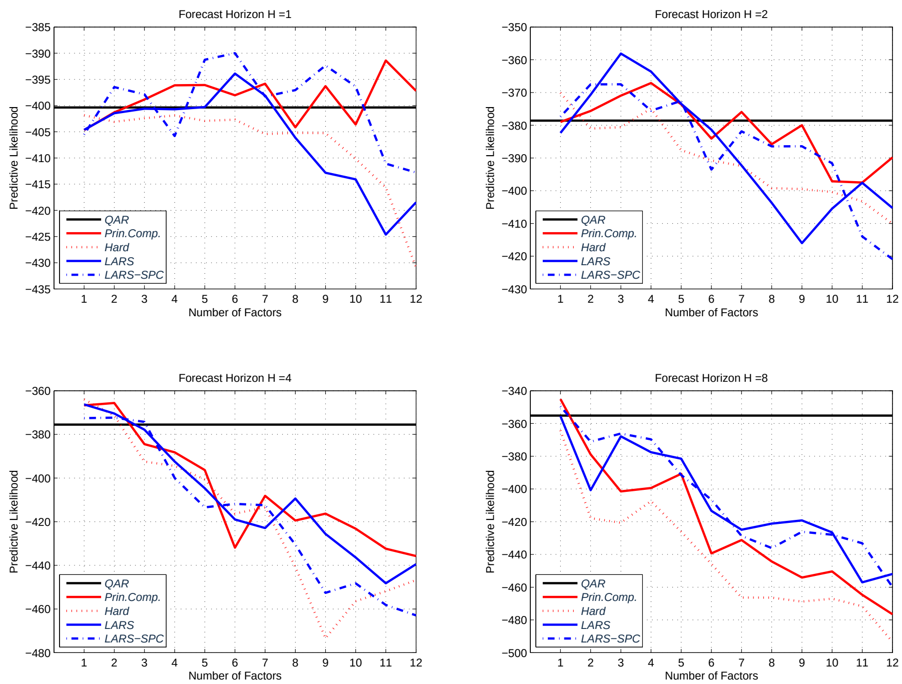


Figure.4.2.b. QAR versus Quantile-FAVAR: Monthly CPI

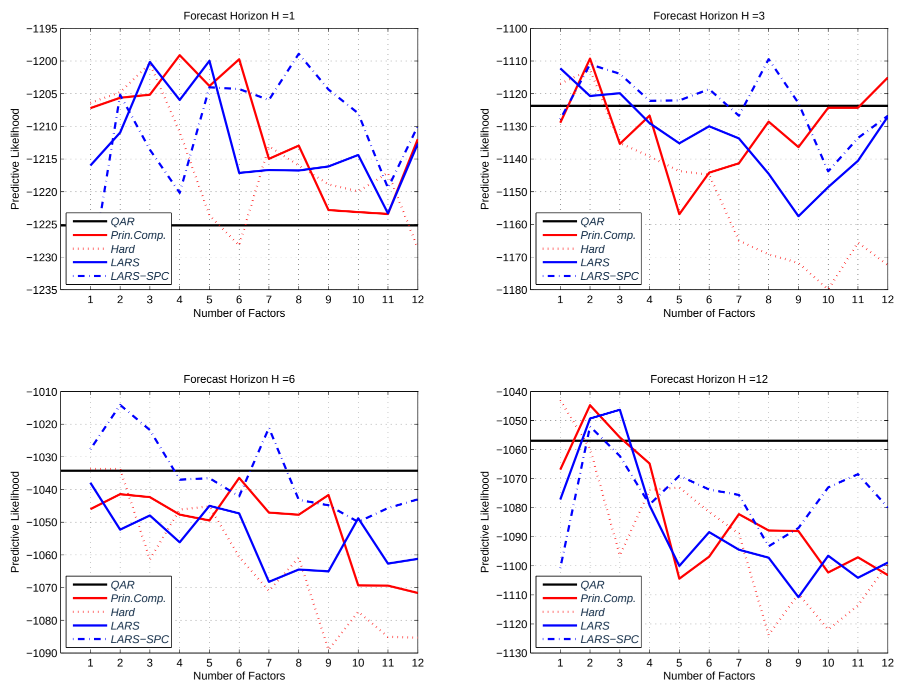


Figure.4.3.a. Semi.Param. AR versus Semi.Param. FAVAR: Quarterly GDP

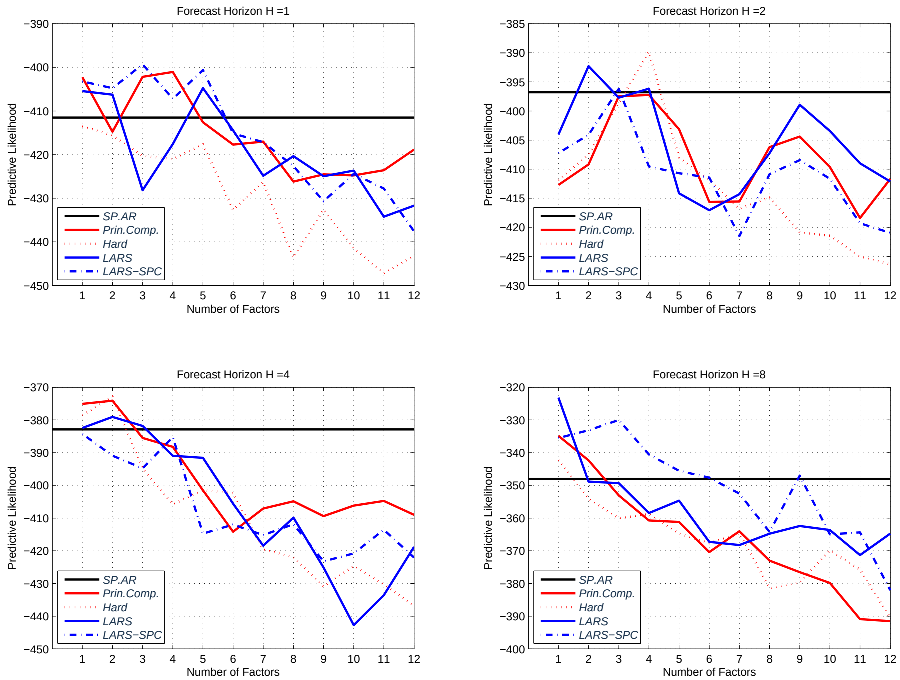


Figure.4.3.b. Semi.Param. AR versus Semi.Param. FAVAR: Monthly CPI

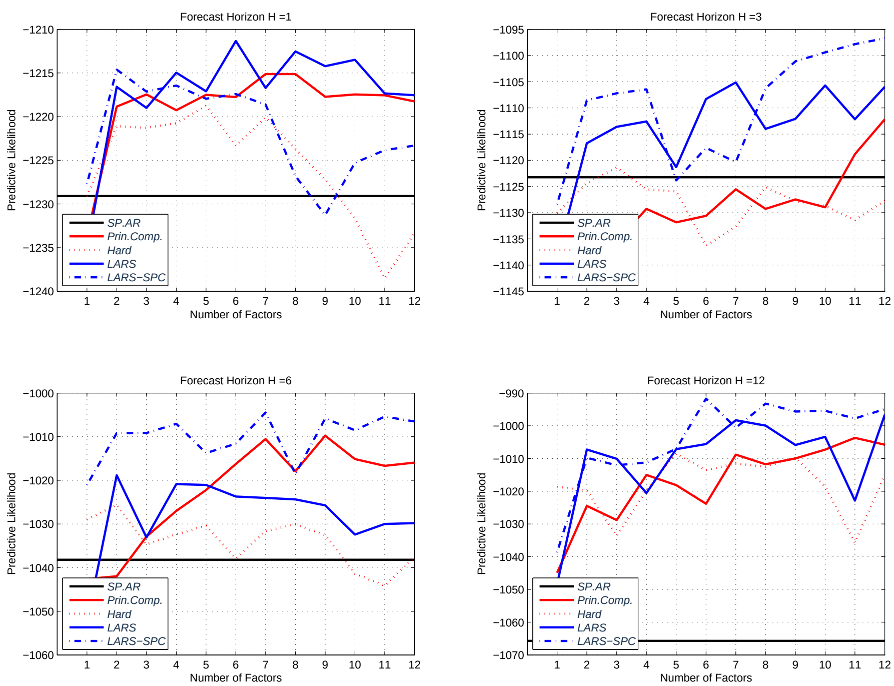


Figure.5.a. Kolmogorov Smirnov Test: Monthly CPI

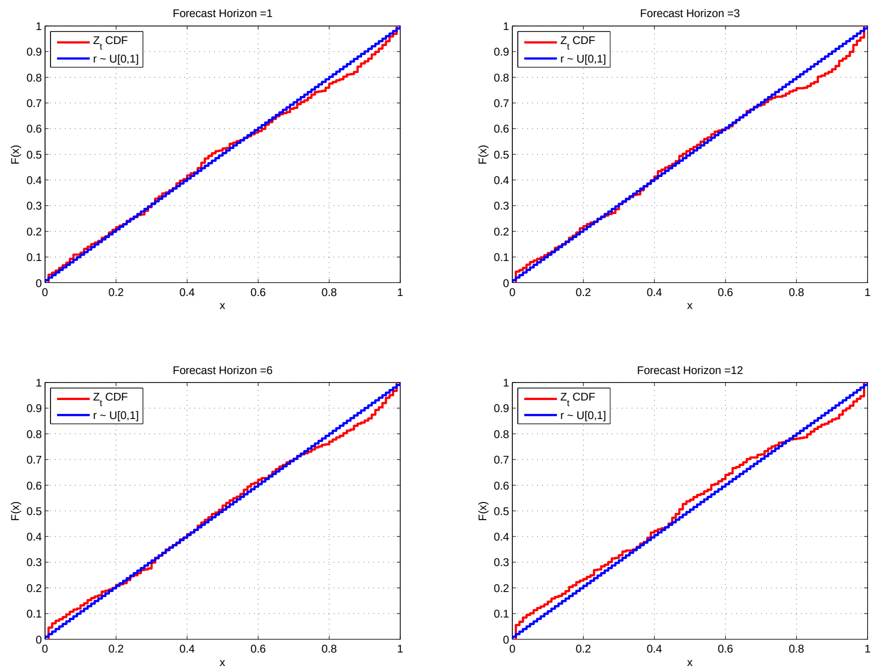
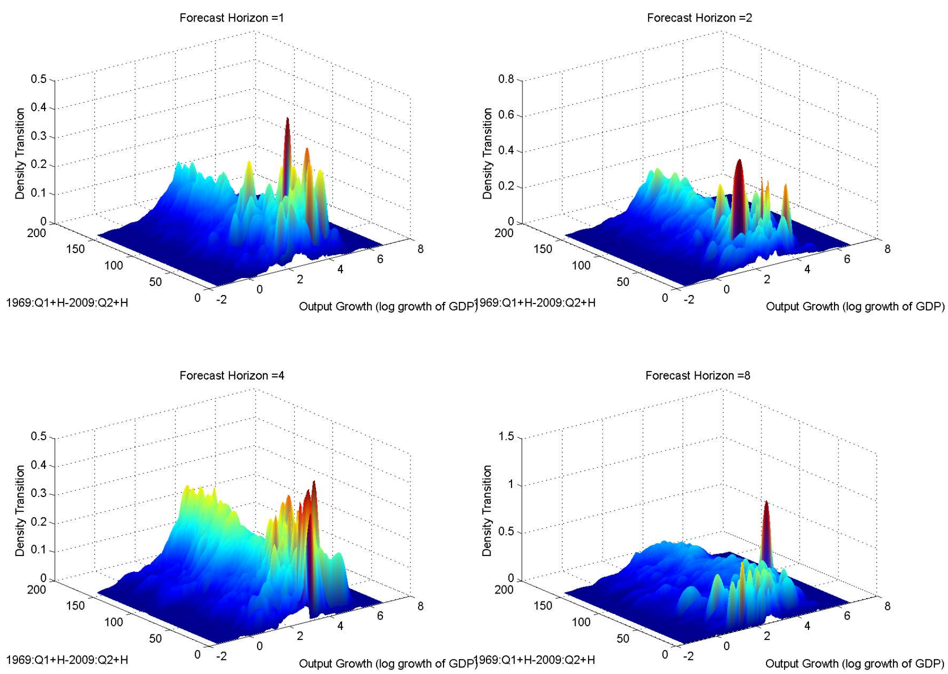


Figure.5.b. Density Transition of Quarterly GDP from 1969:01+H to 2009:06+H



Notes: Enclosed are just two examples. Results from all models can be provided upon request.

Figure.6.a. Horse Race of Predictive Accuracy between QAR and Gaussian AR (monthly CPI)

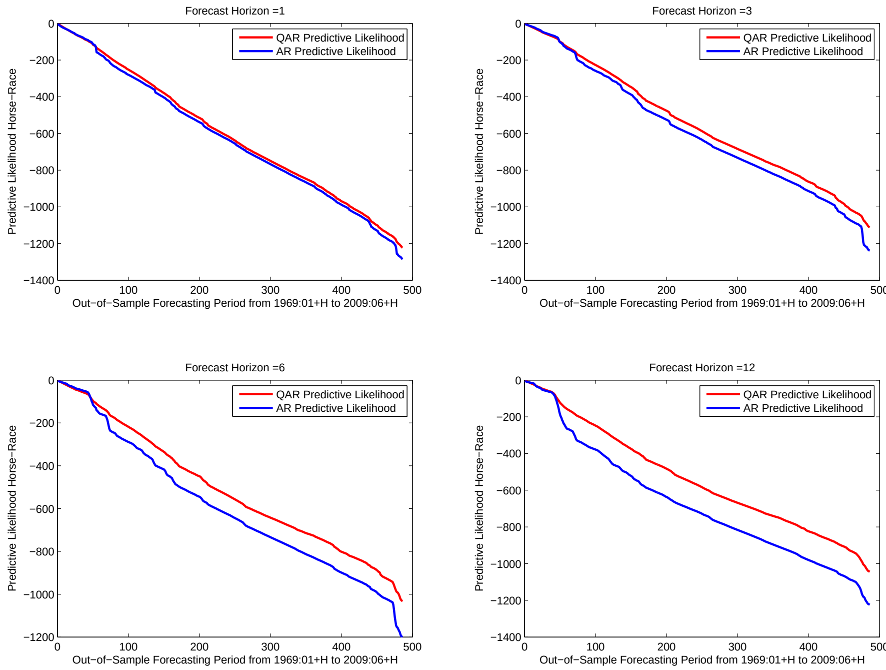
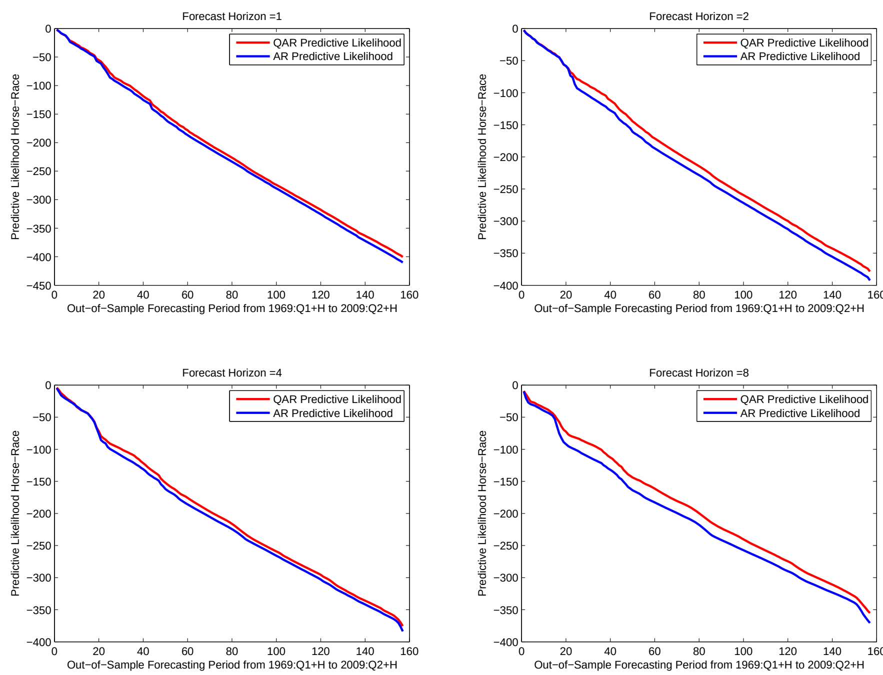


Figure.6.b. Horse Race of Predictive Accuracy between QAR and Gaussian AR (Quarterly GDP)



Notes: Enclosed are just two examples. Results from all models can be provided upon request.