Coalition Formation with Local Public Goods and Group-Size Effect*

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Abstract

Many public goods that are provided by coalitions have a group-size effect. Namely, people prefer to consume a public good in a larger coalition. This paper studies local public goods games with anonymous and separable group-size effect. The core is nonempty when coalition feasible sets are monotonic and players' preferences over public goods satisfy a condition called cardinal connectedness. Moreover, a core allocation consists of connected coalitions.

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1 Introduction

Many public goods are provided by coalitions. For some of these "local" public goods, there is a *group-size effect*, where people prefer to be in a larger coalition given the same public good consumption. The following are some examples: (i) Consumers choose among insurance policies in the market. Having more people under the same policy ensures better risk-sharing. (ii) Political parties promote their policy platforms, and people prefer to join a larger party for a better probability of wining. (iii) Clubs provide

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entertainment to members; people may prefer to socialize with more people sharing some interest. (iv) Professors join academic departments to obtain research resources, and a larger faculty provides better interactions.[1] In the above examples of coalitions, their size also enters into preferences besides public goods. We investigate conditions that guarantee stable formation of coalitions in such games.

Whether stable coalitions can form with nontransferable utility was first studied by Aumann and Drèze (1975). They extend solution concepts which were based on the formation of the grand coalition to coalition structures. Kaneko and Wooders (1982) develop conditions, which are independent of the payoff function, that guarantee the nonemptiness of the core based on Scarf's (1967) balancedness. Le Breton, Owen and Weber (1992) study communication games on graphs where only connected coalitions are effective. When players have preferences over coalition members, the coalition is called a hedonic coalition Drèze and Greenberg (1980) first consider the hedonic aspect where players derive utility from private goods, public goods, as well as coalition members. The literature, then, addresses two issues, coalition provided public goods and pure hedonic coalitions, separately.

Guesnerie and Oddou (1981) and Greenberg and Weber (1986) show the nonemptiness of the core in local public economies where coalitions decide on levels of public expenditures that are financed by taxes. Greenberg and Weber (1993a) and Demange (1994) study models with abstract spaces for public goods. A stronger notion of stability, which looks for core allocations that are also Tiebout equilibria, is obtained. Existence of such stable allocations are shown in the former when preferences are single-peaked on a line, and in the latter when preferences are intermediate on a tree graph. Coalition feasible sets are assumed to be monotonic in both. In these games, additional members enlarge the feasible set of a coalition, but players do not prefer a larger coalition size per se. Pure hedonic coalitions where utility is solely derived from members are investigated by the following authors in games without public goods. Banerjee, Konishi and Sönmez (2001) show that the core may be empty unless restrictive conditions are assumed. Bogomolnaia and Jackson (2002) obtain the existence of individually stable and Nash stable allocations. Iehlé (2005) develops a pivotal balancedness condition that guarantees a core partition. The group-size effect is also studied in strategic form games by Konishi, Le Breton and Weber (1997a). In their game, a coalition is a set of players choosing the same pure strategy, and there are positive externalities from group size. A Nash equilibrium exists when preferences over strategies dominate the group-size effect.

Whether stable coalitions can form in games with both public goods and group-

[1] The group-size effect, in private or public goods, is called the *network effect* in the literature of industrial organization (see, for example, Katz and Shapiro 1985).

size effect is still an open question. We study such a game where the group-size effect can be dominant. We focus on a type of group-size effect that is anonymous and separable; these preferences over coalition size are represented by a common increasing function.[2] Preference structures such as single-peakedness and intermediate preferences are imposed in the public goods literature. These structures link players on a graph (a line or a tree), and require their ordinal preferences to change gradually over this graph. The group-size effect in our model, however, can disrupt these preference structures. When a player's desire for a larger coalition dominates his preferences over public goods, previous results do not hold. Consequently, a stronger structure is required. We use *cardinal connectedness*, which is a preference restriction over public goods only. It stipulates gradual changes in the relative strength of preferences over public goods. Specifically, it imposes a tree graph linking players together such that, for any pair of public goods and for any real number, those players whose utility differences from the two public goods are strictly bigger than the given number form a connected set on the tree. When coalition feasible sets are monotonic and preferences over public goods satisfy cardinal connectedness, the core is nonempty. Moreover, a core allocation consists of connected coalitions.

Section 2 presents the model and results. Section 3 concludes.

2 The model and results

The set of players is denoted by N and the set of public goods by X. Each player $i \in N$ has preferences over X that are represented by a continuous function $u_i : X \to \mathbb{R}$. A *coalition* is a subset $S \subset N$. For each coalition S, there is a set of feasible public goods $\phi(S)$. We call $\phi : 2^N \to 2^X$ a *feasibility correspondence.* A coalition may have an empty feasible set. To eliminate triviality, we assume that there exists $S \subset N$ such that $\phi(S) \neq \varnothing$. The group-size effect is anonymous, separable, and represented by a common function for all players. Utility function $v_i : X \times 2^N$ represents player i's preferences over pairs of a coalition and a public good. If player i consumes public good $x \in X$ in coalition S, his utility is

$$v_i(S, x) = u_i(x) + f(|S|),$$

where $f > 0$ and is strictly increasing. The pair (S, x) is called a coalition–public-good pair. A *local public goods game with group-size effect* $\left(N, X, \phi, (v_i)_{i \in N}\right)$ consists of a

[2] This is a special case of the Drèze and Greenberg (1980) type utility function. Greenberg and Weber (1986) have another application where the utility from group size is derived indirectly through a tax sharing rule.

set of players, a set of public goods, a feasibility correspondence, and a profile of utility functions, where N is finite, X is closed (in its associated topological space), and ϕ is compact-valued.

A *coalition structure* $\Pi \subset 2^N$ is a partition of N such that $\phi(S) \neq \varnothing$ for all $S \in \Pi$. An *allocation* $a : N \to 2^N \times X$ assigns a coalition–public-good pair $a(i)$ to individual i. Allocation a is *feasible* if there is a coalition structure and a list of public goods $\left(\Pi_a, (x_S)_{S \in \Pi_a}\right)$ with $x_S \in \phi(S)$ for all $S \in \Pi_a$ such that $a(i) = (S, x_S)$ for all $i \in S$ and all $S \in \Pi_a$. To simplify notation, we denote $v_i(a(i)) = v_i(S, x_S)$. The feasibility correspondence ϕ is *monotonic* if additional alternatives become feasible to a coalition when it has more members; $\phi(S) \subset \phi(S')$ for all $S, S' \in 2^N$, $S \subset S'$.

Preference structures for public goods, such as single-peakedness and intermediate preferences, are used in the literature. These structures link players on a graph (for example, a line in the former and a tree in the latter), and stipulate ordinal preferences to change gradually on the graph. With the group-size effect, even if preferences have one of the above structures over the public goods space X, the nice structure can be disrupted in the $2^N \times X$ space. This is because a player's desire for a bigger coalition may be larger than his utility difference from public goods. For example, suppose player i prefers x to y in any coalition and the group-size effect has $f(3) - f(2) > u_i(x) - u_i(y)$. Then, i prefers y in a three-person coalition to x in a two-person coalition. Hence, previous structures cannot guarantee a nonempty core, when the group-size effect is not negligible and dominates preferences over public goods in arbitrary ways. Consequently, a stronger restriction is required. We use the following condition that restricts cardinal preferences over public goods.

- Let G be a tree on N. Preferences over public goods satisfy *cardinal connectedness on G* if for any pair $x, y \in X$ and any $t \in \mathbb{R}$, $|t| \leq f(N) - f(0)$, the following set is connected on G: $\{i \in N \mid u_i(x) - u_i(y) > t\}$.

Cardinal connectedness requires that players can be linked on a tree in such a way that, for any pair of public goods and for any real number t, the set of players whose utility differences from the pair of public goods are bigger than t is connected on the tree. It says that players can be linked on a tree according to their preferences: players with similar preferences are connected on the tree, and the strength of preferences changes monotonically when moving along the tree. The number t helps separate players into three sets $\{i \in N \mid u_i(x) - u_i(y) > t\}$, $\{i \in N \mid u_i(x) - u_i(y) = t\}$, and $\{i \in N \mid u_i(x) - u_i(y) < t\}$. For example, suppose $t = 5$. Then the first set, players whose payoffs at x are greater than their payoffs at y plus 5, is a connected set. Notice that the last set is equal to $\{i \in N \mid u_i(y) - u_i(x) > -t\}$. It contains players whose payoffs at y are greater than their payoffs at x minus 5. According to every pair of public goods and every t, there are two mutually exclusive connected sets; each containing players with similar preferences. With t varying, cardinal connectedness stipulates a gradual change of payoffs on the tree G. Due to the common and separable group size effect, the cardinal comparisons among public goods embedded in cardinal connectedness enable comparisons of preferences over public goods and the desirability of a larger coalition.

The Euclidean utility function represents a simple example of this type of preferences: Let $u_i(x) = \|x - a_i\|$ where $x, a_i \in \mathbb{R}$ and constant a_i is player i's ideal point. For $x, y \in X$ and a real number t, the set $\{i \in N \mid u_i(x) - u_i(y) > t\}$ is equivalent to $\{i \in N \mid a_i > (x - y + t)/2\}$ and it is connected on the real line.[3]

A related preference structure is "intermediate preferences" (Grandmont 1978, Demange 1994). It requires that players can be linked on a tree, and for any pair of public goods, the following are two connected sets on the tree: players with the same strict preference and players with the same weak preference. In other words, strict and weak ordinal preferences change gradually over the tree. In contrast, cardinal connectedness requires that the strength of strict cardinal preferences changes gradually over a tree. Connected support (Kung 2006) is a weaker version of intermediate preferences. It requires that for any pair of public goods, the set of players with the same strict preference are a connected set on the tree. These tree structures may represent hierarchical organizations or communication networks in the real world, where players can interact, and which facilitate coalition formation. Cardinal connectedness is based on the same idea that players with similar preferences are connected, but requires a stronger similarity in cardinal preferences.

A feasible allocation a is in the *core* if there is no coalition–public-good pair (S, x) such that $x \in \phi(S)$ and $v_i(S, x) > v_i(a(i))$ for all $i \in S$.

The next example shows the necessity for cardinal connectedness. We present a preference profile that is single-peaked and also satisfies intermediate preferences on public goods. Yet, the core is empty due to the group-size effect. The example is adapted from Example 4.6 in Konishi, Le Breton and Weber 1997a.

Example 1. Let $N = \{1, 2, 3, 4, 5, 6, 7\}$, and $X = \{x, y, z\}$. The feasibility correspondence is constant; $\phi(S) = \{x, y, z\}$ for all S. Players 4 and 5 have the same preferences,

[3] Utility functions used in Demange (1994, p. 50) as examples of intermediate preferences also satisfy cardinal connectedness.

and so do players 6 and 7; $u_4 = u_5$, $u_6 = u_7$. Their utility functions are as follows:

$$\begin{aligned}
u_1(x) &= 2.5, u_1(y) = 4, u_1(z) = 0, \\
u_2(x) &= 0, u_2(y) = 4.5, u_2(z) = 1.7, \\
u_3(x) &= 0, u_3(y) = 0.3, u_3(z) = 0.5, \\
u_4(x) &= 8, u_4(y) = 0.1, u_4(z) = 0, \\
u_6(x) &= 0, u_6(y) = 0.1, u_6(z) = 8,
\end{aligned}$$

with the group-size effect $f(|S|) = |S|$. These preferences are single-peaked when public goods are ordered as $x - y - z$, and they are intermediate preferences when players are linked as $4 - 5 - 1 - 2 - 3 - 6 - 7$.

Players 4 and 5 have strong preferences for x that are not outweighed by the group-size effect. In any core allocation, they belong to the same coalition and consume x. This also means that if any of the remaining players consumes x in a core allocation, they belong to the same coalition as players 4 and 5 because of the group-size effect. For the same reason, players 6 and 7 are in the same coalition and consume z.

We can thus reduce the game for the remaining players 1, 2, and 3, by defining $\hat{x}$ as "consuming x with players 4 and 5 " and $\hat{z}$ as "consuming z with players 6 and 7." The reduced utility functions are

$$\begin{aligned}
\hat{u}_1(\hat{x}) &= 4.5, \hat{u}_1(y) = 4, \hat{u}_1(\hat{z}) = 2, \\
\hat{u}_2(\hat{x}) &= 2, \hat{u}_2(y) = 4.5, \hat{u}_2(\hat{z}) = 3.7, \\
\hat{u}_3(\hat{x}) &= 2, \hat{u}_3(y) = 0.3, \hat{u}_3(\hat{z}) = 2.5,
\end{aligned}$$

with $f(|S|)$ stays the same. The core in the reduced 3-person game with $\{1, 2, 3\}$ and $\{\hat{x}, y, \hat{z}\}$ is identical to the core in the original game (after adding back players 4 to 7). Next, we show that the core is empty.

First, if each player stays alone, each chooses his favorite public good. Yet, they cannot all stay alone since $((1, \hat{x}), (2, y), (3, \hat{z}))$ is blocked by $(1\ 2, y)$. The coalition $(1\ 2\ 3)$ cannot form since then, any choice of public good is blocked by one player due to the cyclic preferences. The remaining candidate allocations are $((1, \hat{x}), (2\ 3, y))$, blocked by $(1\ 2\ 3, y)$, $((1, \hat{x}), (2\ 3, \hat{z}))$, blocked by $(1\ 2, y)$, $((2, y), (1\ 3, \hat{z}))$, blocked by $(1\ 2\ 3, \hat{z})$, $((2, y), (1\ 3, \hat{x}))$, blocked by $(2\ 3, \hat{z})$, $((3, \hat{z}), (1\ 2, \hat{x}))$, blocked by $(1\ 2\ 3, \hat{x})$, and $((3, \hat{z}), (1\ 2, y))$, blocked by $(1\ 3, \hat{x})$. There is no core allocation. ▮

The power of cardinal connectedness requires the assumed common group-size effect. The above example can be modified to show that cardinal connectedness may not guarantee a nonempty core when players do not have a common group-size function.

We adjust utility functions to satisfy cardinal connectedness while keeping preferences over the coalition–public-goods space $2^N \times X$ the same, by varying the group-size effect function individually. Example 2 illustrates that the common group-size effect serves as a normalizing measure that facilitates cardinal comparison across public goods and coalition sizes.

Example 2. The utility functions are modified from Example 1 as follows: $\bar{f}_i = f$ for $i = 1, 2$ and $\bar{f}_i(k) = 6k$ for $i = 3, 4, 5, 6, 7$,

$$\begin{aligned} \bar{u}_1 &= u_1, \bar{u}_2 = u_2, \\ \bar{u}_3(x) &= 0, \bar{u}_3(y) = 5, \bar{u}_3(z) = 5.5, \\ \bar{u}_4(x) &= 50, \bar{u}_6(y) = 5.5, \bar{u}_6(z) = 0, \\ \bar{u}_6(x) &= 0, \bar{u}_6(y) = 5.5, \bar{u}_6(z) = 50, \end{aligned}$$

$\bar{u}_5 = \bar{u}_4$, and $\bar{u}_7 = \bar{u}_6$. Thus, all $\bar{u}_i$ satisfy cardinal connectedness if players are linked as $4 - 5 - 1 - 2 - 3 - 6 - 7$.[4] Notice that $\bar{u}_i + \bar{f}_i$ defines the same preferences over $2^N \times X$ space as in Example 1. See player 3 for example, the payoff increase from one additional player in the coalition outweighs any payoff difference in public goods. Thus, there is no core allocation. ∎

Theorem 1. *When preferences satisfy cardinal connectedness on G and feasible sets are monotonic, a local public goods game with group-size effect has a nonempty core.*

Proof. An algorithm that constructs a core allocation is defined as follows.[5] For convenience, we temporarily assign a null public good μ to coalitions that have empty feasible sets, and $u_i(x) > u_i(\mu) + f(|N|)$ for all $x \in X$ and all $i \in N$. Thus, μ is the least preferred public good for all players. Let $X' = X \cup \{\mu\}$. We will show later that the final construction does not involve μ.

Take $r \in N$ to be the root of tree G. Rooted tree G^r assigns priorities to players as follows. The distance between player i and r is $\delta(r, i) = k$ if r and i are linked by a path of length k. We say that i has priority $(k+1)$. Let $\bar{k} = \max_{i \in N} \delta(r, i)$ be the

[4] This is easily checked with the following table of utility differences.

players	4,5	1	2	3	6,7
$\bar{u}_i(x) - \bar{u}_i(y)$	44.5	-1.5	-4.5	-5	-5.5
$\bar{u}_i(y) - \bar{u}_i(z)$	5.5	4	2.8	-0.5	-44.5
$\bar{u}_i(x) - \bar{u}_i(z)$	50	2.5	-1.7	-5.5	-50

[5] Similar versions of algorithms that utilize an ordering of agents can be found in Greenberg and Weber (1993b), Demange (2004) and Kung (2006).

maximal length on G^r. Let N^i denote the subtree originating from i that contains i and players with lower priorities. Note that $N^i \cap N^j = \varnothing$ if i, j have the same priority.

Let

$$\tilde{N}^i = \left\{ (S, x) \in 2^{N^i} \times X \mid x \in \phi(S),\ i \in S,\ S \text{ is connected} \right\}.$$

It is the set of coalition–public-good pairs (S, x) that are feasible, and whose coalitions S are connected and composed of i and only other players on i's subtree. Let $\bar{v}_i = v_i(S^*, x^*)$, where $(S^*, x^*) \in C^i$, denote the maximized utility for i in C^i.

Next, we define the *top choice set* C^i for player i recursively. Suppose C^j are defined for all j on i's subtree N^i. Then

$$C^i = \arg\max \left\{ v_i(S, x) \ \ s.t.\, (S, x) \in \tilde{N}^i \text{ and } v_j(S, x) \geq \bar{v}_j, \forall j \in S \backslash i \right\}.$$

It is the set of utility maximizing coalition–public-good pairs (S, x) for i among those that are in $\tilde{N}^i$ and give each other coalition member $j \in S \backslash i$ his maximal utility $\bar{v}_j$ in C^j.

To sum up, C^i consists of player i's most preferred coalition–public-good pairs among all such feasible pairs that consist of connected coalitions (containing i) on i's subtree N^i, and keep coalition members no worse off than at their top choice sets. The next lemma shows that the top choice set is well-defined.

Lemma 1. *$C^i \neq \varnothing$ for all $i \in N$.*

Proof. First, let $R_i(S, \bar{v}_i) = \{(T, x) \in \{S\} \times X \mid x \in \phi(T)\ ,\ v_i(T, x) \geq \bar{v}_i\}$ denote i's upper contour set composed of coalition S and supporting utility no less than $\bar{v}_i$. Let

$$D^i = \bigcup\nolimits_{\left\{S \mid (S,x) \in \tilde{N}^i\right\}} \left(\cap_{j \in S \backslash i} R_j(S, \bar{v}_j) \cap \{(S, x) \mid x \in \phi(S)\} \right).$$

Thus, D^i is the set of all feasible coalition–public-good pairs that consist of connected coalitions on i's subtree N^i containing i, and keep coalition members no worse off than at their top choice sets. The set C^i consists of i's most preferred pairs in D^i. Since player i can always remain by himself, $D^i \neq \emptyset$. All $R_i(S, .)$ and $\phi(S)$ are compact, and the set $\left\{ S \mid (S, x) \in \tilde{N}^i \right\}$ is finite. Thus, D^i is the union of finitely many compact sets. Since v_i is continuous, $C^i \neq \varnothing$. ∎

In the following, we construct an allocation $\hat{a}$ recursively using top choice sets. Given a collection of pairs $\left\{ \left(S^i, x^i\right) \right\}_{i \in N}$ such that $\left(S^i, x^i\right) \in C^i$ for all $i \in N$, we assign coalition–public-good pairs sequentially, starting from r. Let $L^0 = \{r\}$.

$\hat{a}(i) = (S^r, x^r)$ for all $i \in S^r$. Let $L^1 = \{j \in N \backslash S^r \mid \nexists h \in N \backslash S^r\ s.t.\, \delta(r, h) < \delta(r, j)\}$.

$\hat{a}(i) = \left(S^j, x^j\right)$ for all $i \in S^j$ and all $j \in L^1$.

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Suppose $\hat{a}(i)$ is assigned for all $i \in S^j$ and all $j \in L^{m-1}$.

Let $\hat{S}(m-1) = \bigcup_{i \in \cup_{k=0}^{m-1} L^k} S^i$ and

$$L^m = \left\{ j \in N \backslash \hat{S}(m-1) \mid \nexists h \in N \backslash \hat{S}(m-1) \ \ s.t. \delta(r,h) < \delta(r,j) \right\}.$$

$\hat{a}(i) = \left(S^j, x^j\right)$ for all $i \in S^j$ and all $j \in N \backslash L^m$.

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Since N is finite, there is an integer $\bar{l} \leq \bar{k} - 1$ such that $L^{\bar{l}+1}(r) = \varnothing$.

Note that every coalition that has been assigned is connected on G and adjacent to another coalition. Let $L = \bigcup_{k=0,\ldots,\bar{l}} L^k(r)$. Thus, $\left\{S^i\right\}_{i \in L}$ is a partition of N. The collection of pairs $\left\{\left(S^i, x^i\right)\right\}_{i \in L}$ constitute allocation $\hat{a}$.

Lemma 2. *$v_i(\hat{a}(i)) \geq \bar{v}_i$ for all $i \in N$.*

Proof. For all $i \in N$, either $i \in L$ and $\hat{a}(i) = \left(S^i, x^i\right) \in C^i$, or $i \notin L$, $i \in S^j$ for some $j \in L$, then, $\hat{a}(i) = \left(S^j, x^j\right)$ and $v_i\left(S^j, x^j\right) \geq \bar{v}_i$. ∎

Next, we show that $\hat{a}$ does not involve the null public good μ.

Lemma 3. *For all $i \in N$, $\hat{a}(i) \neq (S, \mu)$ for any $S \subset N$.*

Proof. Suppose there is a coalition S that consumes μ. Suppose there is a coalition T that is adjacent to S and T consumes a public good $x \neq \mu$. We will show that every T adjacent to S also consumes μ.

Suppose ij is the linking edge of S and T, and $i \in S$, $j \in T$. First, suppose i has a higher priority than j. Thus, $T \subset N^i$ and for all $h \in T$, $v_h(T \cup \{i\}, x) > v_h(\hat{a}(h)) \geq \bar{v}_h$. By monotonicity, $x \in \phi(T \cup \{i\})$. Therefore, $(T \cup \{i\}, x) \in D^i$ (as defined in Lemma 3). Since $(S, \mu) = \hat{a}(i)$, Lemma 3 implies $v_i(S, \mu) > \bar{v}_i > v_i(T \cup \{i\}, x)$. This is a contradiction.

Second, suppose j has a higher priority than i. Suppose g is the player with the highest priority in T (this g is unique). Thus, $(T, x) = (S^g, x^g) \in C^g$. Then $v_h(T \cup S, x) > v_h(T, x) \geq \bar{v}_h$ for all $h \in T$ and $v_h(T \cup S, x) > v_h(S, \mu) \geq \bar{v}_h$ for all $h \in S$. By monotonicity, $x \in \phi(T \cup S)$. This means $(T \cup S, x) \in D^g$ and $(T, x) \notin C^g$; a contradiction.

Since every coalition is adjacent to another, all coalitions consume μ. Note that there exists $S \in 2^N$ such that $\phi(S) \neq \varnothing$. So, there is $x \in \phi(N)$ and $x \neq \mu$. Moreover, $v_i(N, x) > v_i(\hat{a}(i)) \geq \bar{v}_i$ for all $i \in N \backslash r$. This means $(N, x) \in D^r$; a contradiction. ∎

So far, we have shown that $\hat{a}$ is well-defined. Next, we introduce some characterizing properties of the core, which aid the proof.

A feasible allocation a satisfies the *separation property on* G if for any linking edge ij of two adjacent coalitions on a tree G,

$$v_h(a(i)) \geq v_h(a(j)) \text{ for all } h \in M(ij),$$
$$v_h(a(j)) \geq v_h(a(i)) \text{ for all } h \in M(ji),$$

where $M(ij) = \{h \in N \mid ij \notin p(i,h)\}$, and $p(ij)$ is the path linking i and j on G.

Lemma 4. *In a local public goods game with group-size effect where preferences satisfy cardinal connectedness on* G *and feasible sets are monotonic, allocation* a *is in the core if (i)* a *is not blocked by any coalition that is connected on* G*, and (ii)* a *satisfies the separation property on* G*.*

Proof. First, the separation property leads to $v_i(a(i)) \geq v_i(a(j))$ for all $i, j \in N$. In allocation a, any two players $i, j \in N$ are linked on G by a unique path that passes through adjacent coalitions. Let $i_0 = i$, $i_k = j$ and $p(i,j) = \{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$. For all $m = 1, ..., k$, either i_{m-1} and i_m belong to the same coalition and $a(i_{m-1}) = a(i_m)$, or $i_{m-1} i_m$ links two adjacent coalitions and $i \in M(i_{m-1} i_m)$, which means $v_i(a(i_{m-1})) \geq v_i(a(i_m))$. So, $v_i(a(i_0)) \geq v_i(a(i_k))$.

We will show that no coalition can block. Suppose S blocks with x and S is not connected. Let T be the minimal connected set containing S. That is, $S \subset T \in 2^N$ and there is no connected $T' \in 2^N$, $T' \neq T$ such that $S \subset T' \subset T$. For all $h \in T \backslash S$, we can find $i, j \in S$ such that h is on the path linking i and j. Denote $a(h) = (S', y)$. We have $v_i(S, x) > v_i(a(i)) \geq v_i(S', y)$ and $v_j(S, x) > v_j(a(j)) \geq v_j(S', y)$. Since the set $\{k \in N \mid u_k(x) - u_k(y) > f(|S'|) - f(|S|)\}$ is connected, we have $v_h(S, x) > v_h(S', y)$ as well. Also, since T is larger than S, $v_i(T, x) > v_i(S, x)$ for all $i \in T$. Moreover, by monotonicity, $x \in \phi(T)$. Thus, T is a connected coalition that blocks a. This is a contradiction. ∎

Finally, we show that $\hat{a}$ satisfies the above two properties.

Lemma 5. *The allocation* $\hat{a}$ *is not blocked by any connected coalition on* G*, and satisfies the separation property on* G*.*

Proof. (i) Suppose there is a pair (S,x) such that $x \in \phi(S)$, $v_i(S,x) > v_i(\hat{a}(i))$ for all $i \in S$, and S is connected. Then, $v_i(S,x) > v_i(\hat{a}(i)) \geq v_i\left(S^i, x^i\right)$ for all $i \in S$. Suppose g is the player of the highest priority in S; then $S \in N^g$, $(S,x) \in D^g$, and $v_g(S,x) > v_g(S^g, x^g)$. This is a contradiction.

(ii) Since $\hat{a}$ consists of connected coalitions, there is a unique edge ij linking two adjacent coalitions S and T. Suppose $i \in S$, $j \in T$, S consumes x, and T consumes y. Without loss of generality, suppose i has a higher priority than j. First, $y \in \phi(T \cup \{i\})$ and $(T \cup \{i\}, y) \in D^i$, so $v_i(\hat{a}(i)) \geq v_i(T \cup \{i\}, y) > v_i(\hat{a}(j))$ because of $\hat{a}(i) \in C^i$ and the group-size effect. Second, suppose $v_j(\hat{a}(i)) \geq v_j(\hat{a}(j))$; then, $v_j(\hat{a}(i)) \geq \bar{v}_j$. By monotonicity, $x \in \phi(S \cup \{j\})$. Let g be the player of the highest priority in S. Then, $(S \cup \{j\}, x) \in D^g$ which means $(S,x) \notin C^g$; a contradiction. So, $v_j(\hat{a}(j)) > v_j(\hat{a}(i))$. Finally, suppose there is $h \in M(ij)$ such that $v_h(\hat{a}(j)) > v_h(\hat{a}(i))$. Then $j,h \in \{k \in N \mid u_k(y) - u_k(x) > f(|S|) - f(|T|)\}$. Since i is on the path linking j,h, the above set is not connected and this violates cardinal connectedness. So, there is no $h \in M(ij)$ such that $v_h(\hat{a}(j)) > v_h(\hat{a}(i))$. By the same argument, there is no $h \in M(ji)$ such that $v_h(\hat{a}(i)) > v_h(\hat{a}(j))$. ∎

And this concludes that $\hat{a}$ is in the core. ∎

Proposition 1. *When preferences satisfy cardinal connectedness on G and feasible sets are monotonic, all core allocations in a local public goods game with group-size effect consist of connected coalitions.*

Proof. First, we show that a core allocation a satisfies the separation property. Take a linking edge ij of two coalitions. Suppose $a(i) = (S,x)$ and $a(j) = (T,y)$. Suppose $v_j(S,x) \geq v_j(T,y)$. Then, by group-size effect, $v_j(S \cup \{j\}, x) > v_j(T,y)$ and $v_i(S \cup \{j\}, x) > v_i(T,y)$. Moreover, by monotonicity, $x \in \phi(S \cup \{j\})$. This means that a is not in the core. This is a contradiction. So, $v_j(T,y) > v_j(S,x)$. By the same argument, $v_i(S,x) > v_i(T,y)$. Suppose there is $h \in M(ij)$ such that $v_h(T,y) > v_h(S,x)$, then $j,h \in \{k \in N \mid u_k(y) - u_k(x) > f(|S|) - f(|T|)\}$. Since i is on the path linking j,h, the above set is not connected, and this violates cardinal connectedness. So, there is no $h \in M(ij)$ with $v_h(a(j)) > v_h(a(i))$. By the same argument, there is no $h \in M(ji)$ with $v_h(a(i)) > v_h(a(j))$. Note that the separation property implies $v_i(a(i)) \geq v_i(a(j))$ for all $i,j \in N$ as in the proof of Lemma 4.

Suppose there is a coalition $S \in \Pi_a$ which is not connected. Let T be the minimal connected coalition containing S. Let $i \in T \backslash S$. Note that all $i \in T \backslash S$ is on a path linking two players in S. Then, by group-size effect and the separation property, $v_j(T, x_S) >$

$v_j(S, x_S) \geq v_j(a(i))$ for all $j \in S$ and all $i \in T \backslash S$. Denote $a(i) = (S', y)$. Since the set $\{k \in N \mid u_k(x_S) - u_k(y) > f(|S'|) - f(|S|)\}$ is connected, $v_i(S, x_S) > v_i(a(i))$ for all $i \in T \backslash S$. Moreover, by group-size effect, $v_i(T, x_S) > v_i(a(i))$ for all $i \in T \backslash S$. Finally, by monotonicity, $x_S \in \phi(T)$. So, T blocks with x_S, in contradiction with a being in the core. ▮

3 Concluding Remarks

Drèze and Greenberg (1980) proposed a general form of utility functions that contains public and private goods and coalition members. Subsequent works investigate public good models and pure hedonic models separately. We bring these two features together in a simple hedonic utility function and focus on the group-size effect of public goods. Players have preferences over coalition size as well. We derive characterizing properties for the core and obtain its nonemptiness. With group-size effect, each coalition is composed of players with similar preferences; they are connected on a tree graph. Moreover, in a core allocation, every player prefers his own coalition to any other. Our model incorporates positive group-size effect with public goods in coalition formation. The preference structure we use is based on cardinal preferences and has not been used in earlier literature. We obtain stronger results than Demange (1994) and Kung (2006), which consider a version of the game that exhibits no group-size effect.

While positive group-size effects have been studied in the coalition formation literature, the case of negative group-size effects, namely, congestion, has attracted little attention. Jackson and Nicolò (2004) and Bogomolnaia and Nicolò (2005) present axiomatic studies of rules to assigning players to one and two public facilities with congestion. Milchtaich (1996) and Konishi, Le Breton and Weber (1997c) investigate congestion in strategic form games. The only article we found on endogenous coalitions is Konishi, Le Breton and Weber (1997b), which shows the existence of *free mobility equilibrium*, where no player deviates unilaterally by joining another jurisdiction or staying alone, for a local public goods economy with congestion. Such equilibrium exists when all individuals have a common congestion function, utility is quasi-linear in the private good, and local public goods are financed by a poll tax. Further results concerning, for example, other equilibrium concepts, alternative taxation or abstract feasible sets, and more general preferences, are still open research questions.

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