

Equity Returns and Business Cycles in Small Open Economies

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June 25, 2009

Abstract

This is the first paper in the literature to match key business cycle moments and long-run equity returns in a small open economy with production. These results are achieved by introducing three modifications to a standard real business cycle model: (1) borrowing and lending costs are imposed to increase the volatility of the intertemporal marginal rate of substitution; (2) investment adjustment costs are assumed to make equity returns more volatile; and (3) GHH preferences are employed to smooth consumption. We also decompose the contributions of productivity, the world interest rate, and government expenditure shocks to the equity premium. Our results are based on data from Argentina, Brazil, and Chile.

JEL Classification: E32; E44; F41; G12; G15.

Keywords: Asset Pricing; Equity Returns; Dynamic Stochastic General Equilibrium Model; Real Business Cycle; Small Open Economy.

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[§]We would like to thank seminar participants at Duke University and UNC Chapel Hill. We are grateful for comments by Eric Aldrich, Tim Bollerslev, Neville Francis, A. Ronald Gallant, Lutz Hendricks, Bruce Mizrach, George Tauchen, and Michael Salemi.

1 Introduction

The equity premia in some small open economies are quite high. For example, the equity premia in Argentina and Brazil are, respectively, 12.72% and 19.68%. However, the literature on equity premia in small open economies with production is rather thin in the two and a half decades following the seminal work of Mehra and Prescott (1985). The difficulty in generating the long-run equity premium in such an economy is that the exogenous world interest rate, as one important driving force, is quite smooth. As a result, the intertemporal marginal rate of substitution (IMRS) is not volatile enough. Since a volatile IMRS is required to replicate high equity premia, simply extending, for example, either the Jermann (1998) or Boldrin et al. (2001) models from a closed economy to a small open economy fails to achieve what these models accomplish in the closed economy case. More specifically, in a small open economy characterized by those models, large changes in business cycle moments have only a trivial effect on equity premia. To handle this problem in light of the smooth world interest rate issue, we impose borrowing and lending costs to make the IMRS more volatile.

By introducing borrowing and lending costs, we break down the direct link between the volatility of the world risk-free rate and the volatility of the IMRS. This mechanism makes the supply of debt inelastic and forces consumption to become more sensitive to exogenous shocks. Thus, with these new costs, we magnify the volatility of the IMRS by linking it to the borrowing and lending margin alongside the exogenous world interest rate. Under this condition adjustment costs generate sufficiently high volatility of equity returns, while use of the Greenwood et al. (1988) (henceforth GHH) utility function depresses the volatility in generated consumption. As a result, with three modifications, GHH preferences, capital adjustment costs, and borrowing and lending costs, we are able to match key business cycle moments and long-run equity returns as observed in the data. This is the first paper in the literature to do so in a small open economy with production. As such, our model is a suitable vehicle in which to carry out policy analysis since it satisfies what Barro (2009) calls the “Atkeson-Phelan principle,” after Atkeson and Phelan (1994), in that it replicates the way small open economies price consumption uncertainty.

Our work is related to three strands of the existing literature. First, it builds upon work done on the equity premium puzzle in a closed economy. The models in this literature range from consumption-based to production-based asset pricing models. Consumption-based asset pricing models employ various types of preferences. Mehra and Prescott (1985) were the first to show that the equity premium puzzle cannot be explained under constant relative risk aversion (CRRA) utility since the consumption profile, based on historical data,

is quite smooth.

Campbell and Cochrane (1999) explained equity premia by linking asset prices to deviations of consumption from an external habit. The mechanism is as follows. In comparison to CRRA utility, as consumption reverts towards habit in business cycle troughs, the curvature of the habit formation utility function rises more sharply, which causes asset prices to fall and expected returns to rise more, accordingly. As a result, with habit formation, even though consumption is smooth, the IMRS can be quite volatile.

With Epstein and Zin (1991) preferences, the coefficient of risk aversion and the intertemporal elasticity of substitution (IES) are separated. As a result, the equity premium is not only a function of the consumption profile. It is also a function of volatile consumption-delivering portfolio returns; see Bansal and Yaron (2004) and Bansal (2008), among others. In an endowment economy, this separation between the coefficient of risk aversion and the IES is sufficient to generate the equity premium.

However, altering the utility function alone is not sufficient to explain equity premia in a production economy. As shown in Jermann (1998) and Boldrin et al. (2001), the reason for this failure is the relatively low volatility of the rate of return on equity generated by the model. To explain the equity premium, it is necessary to boost this volatility. This can be accomplished by introducing investment adjustment costs. However, this innovation will result in a higher volatility of consumption growth in a CRRA setting. Introduction of habit formation depresses the volatility of the generated consumption, and hence reconciles the results with smooth historical consumption data. In a closed production economy, both Jermann (1998) and Boldrin et al. (2001) are able to match the moments of business cycles and equity premia with investment adjustment costs and habit formation preferences. Without adjustment costs in a similar setting, Constantinides and Duffie (1996) and Tallarini Jr. (2000) are not successful in doing this. The major shortcoming of all these DSGE models of asset pricing is the “risk-free rate volatility problem”: to explain the equity premium it is necessary to increase the variation in the IMRS, which results in highly volatile risk-free rates that are at odds with the observed data.

This paper is also motivated by the literature on asset pricing in a small open economy. Mendoza and Smith (2006) is an important representative of such work. They focus, however, on the short-run dynamics of the equity premium due to potential sudden stops. Accordingly, their model cannot generate the equity premium in the long run since sudden stops are very rare events. In comparison, our research is concerned with the long-run dynamics of equity premia. We believe our long-run emphasis is warranted since, as noted above, the data show that long-run equity premia are huge for some arguably important small open

economies. Further, heretofore no mechanism has been provided in the literature to explain these observed long-run equity premia. Our analysis shows that, for a small open economy with production, to replicate equity premia it is not sufficient to use GHH preferences and impose investment adjustment costs. We need to also assume borrowing and lending costs. This is due to the behavior of the world risk-free interest rate, which is one of the most important driving forces in a small open economy. As noted above, its observed volatility is quite low. In a general equilibrium model, this property of world interest rates prevents the volatility of the IMRS from being high enough to produce the long-run equity premia found in the data in the absence of borrowing and lending costs.

Our research is also related to the literature on world interest rate and government expenditure shocks. As Mendoza (1991), Neumeyer and Perri (2005), and Uribe and Yue (2006) show, shocks to world interest rates are important in driving business cycles in small open economies. In our study, we use data from three Latin American emerging economies: Argentina, Brazil, and Chile. As documented in Bekaert and Harvey (2003), among other studies, government expenditures, especially in the wake of fiscal imbalances, affect risk premia in emerging markets. Moreover, Burnside et al. (2004) and Lubik and Schorfheide (2006) demonstrate that government expenditure shocks have a significant impact on business cycles. In the case of Argentina especially, fiscal policy shocks have been identified as destabilizing macroeconomic factors for quite some time. As a result, we include both world interest rate and government expenditure shocks in our analysis.

The rest of the paper is organized as follows. Section 2 presents the benchmark economy and defines the equilibrium. We discuss the data used and the calibration procedure in Section 3, and in Section 4 we present both our equity return and business cycle moment results. Section 5 concludes.

2 Benchmark Economy

We study a one sector economy in the benchmark model. This economy has three types of agents: the representative domestic household, firms, and the government. A joint exogenous stochastic process of productivity, the world interest rate, and government expenditures drives the economy. We assume that in this economy, the government does not invest or produce any goods and services. It collects a lump-sum tax to finance its expenditures.

2.1 The Representative Household

The representative household chooses hours and consumption to maximize lifetime expected utility given the budget constraint. The household receives profits, capital rents, and labor income from the firms. There are two means to smooth consumption: purchase of one-period international non-state contingent real bonds, and investment. The model has two real frictions: investment adjustment costs as in Mendoza (1991), and borrowing and lending costs for the household as in Uribe and Yue (2006). Formally, the representative household maximizes life-time expected utility:

$$\max_{\{c_t, h_t, i_t, k_{t+1}, d_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t), \quad (2.1)$$

$$\theta_0 = 1, \theta_{t+1} = \beta(\tilde{c}_t, \tilde{h}_t)\theta_t, t \geq 0,$$

where \mathbb{E}_0 denotes the expectation operator conditional on information available at time $t = 0$. The variables θ_t , c_t , h_t , i_t , k_{t+1} , and d_t denote the subjective discount factor, consumption, hours, investment, capital, and net foreign debt position, respectively. The variables \tilde{c}_t and \tilde{h}_t denote the cross-sectional averages of consumption and labor supply, respectively, which the individual households take as given.¹

We assume consumers have GHH preferences. The functional forms of the GHH utility function and the subjective discount factor are given by:

$$U(c, h) = \frac{(c - \frac{h^\omega}{\omega})^{1-\gamma} - 1}{1-\gamma},$$

$$\beta(c, h) = \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\beta_1},$$

where γ is the coefficient of risk aversion, and the IES can be shown to be approximately equal to $\frac{c-h^\omega/\gamma}{\gamma c}$.² As long as $\beta_1 < \gamma$, these preferences guarantee that there exists a unique limiting distribution of state variables, and that the consumption good in every period is a normal good; see Mendoza (1991). The suitability of GHH utility for dynamic programming follows from Epstein (1983). It can be shown that, as with Epstein-Zin and in contrast to CRRA preferences, GHH utility allows for separation of the coefficient of risk aversion and

¹The cross-sectional means \tilde{c}_t and \tilde{h}_t are used to simplify computation of the representative household's optimal choice. Schmitt-Grohé and Uribe (2003) showed that use of either \tilde{c}_t and \tilde{h}_t or c_t and h_t leads to almost identical impulse response functions.

²The derivation of the IES is shown in the Appendix.

the IES, i.e., knowledge of γ is not sufficient to determine the value of the IES. Another feature of GHH utility is that it rules out the wealth effect on the labor supply decision.

Following Mendoza (1991), we modify the GHH utility function such that the household endogenizes its subjective discount rate of time preference. Let consumption be a composite of final good consumption and the disutility of labor supply. Then the endogenous subjective discount factor is decreasing in past consumption. As a result, GHH utility implies that mean-reverting behavior exists in consumption. Whenever the representative household changes its current consumption, both the marginal utility of current consumption and the impatience level for future consumption change. Specifically, an increase in current consumption causes a decrease in marginal utility. All else equal, this implies a comparable increase in future consumption. However, the subjective discount factor decreases as well, which *ceteris paribus* leads to a decrease in future consumption. Together, this means that future consumption will not increase as much as today's consumption does, indicating reversion towards the mean.

Endogenizing the subjective discount factor is one way to modify the standard real business cycle model to assure stationary behavior, as noted in Schmitt-Grohé and Uribe (2003). It is important to point out, however, that this modified GHH utility is not suitable for incorporation of balanced growth as studied in, for example, Boldrin et al. (2001) and Aguiar and Gopinath (2007).³ Accordingly, we do not consider the effects of economic growth in our model; our focus is on business cycle fluctuations.

In each period the representative household is subject to the budget constraint:

$$d_t + w_t h_t + r_t k_t \geq r_{t-1}^f d_{t-1} + \Psi(d_t - \bar{d}) + c_t + i_t + \Gamma_t + \Phi(k_{t+1} - k_t). \quad (2.2)$$

The variables r_t and w_t denote, respectively, the return on capital and the wage rate. The variable r_t^f is the world risk-free rate from period t to $t + 1$. We discuss the dynamics of this variable in detail in our treatment of the driving force. The variable Γ_t denotes the government lump-sum tax, and the variable k_t is physical capital. Its law of motion is given by:

$$k_{t+1} = (1 - \delta)k_t + i_t. \quad (2.3)$$

We assume all equity is held by domestic households. In this economy, dividends are returns to ownership of capital, which includes physical capital and intangible assets such as patents.

³If we assume that the subjective discount factor is constant, then balanced growth is admissible.

Equity is assumed to be equivalent to the capital stock.

$\Psi(d_t - \bar{d})$ denotes borrowing costs and \bar{d} is the non-stochastic steady-state value of the debt position. In the absence of borrowing and lending costs, households can borrow or lend at the world interest rate freely. Practically, this is not a tenable assumption for most small open economies. Only a handful of these countries have access to world debt markets with trivial costs. We posit that households make their decisions based on the “effective interest rate,” i.e., the interest rate faced by the households in a small open economy is usually equal to world interest rate plus a markup. In our model this markup is a function of the debt position of the economy. We follow Schmitt-Grohé and Uribe (2003) and Aguiar and Gopinath (2007) by defining borrowing and lending costs indirectly as:

$$1 - \Psi'(d_t - \bar{d}) = \frac{1}{1 + \psi [\exp(d_t - \bar{d}) - 1]}, \quad (2.4)$$

where $\psi > 0$.

$\Phi(k_{t+1} - k_t)$ represents investment adjustment costs. As shown by Kydland and Prescott (1982) and Jermann (1998), investment adjustment costs are important factors in explaining business cycle movements and the equity premium. Following Mendoza (1991), we include such costs in our model and assume its functional form to be:

$$\Phi(k_{t+1} - k_t) = \frac{\phi}{2} (k_{t+1} - k_t)^2. \quad (2.5)$$

The representative household is subject to the non-Ponzi-game condition:

$$\lim_{j \rightarrow \infty} \mathbb{E}_t \frac{d_{t+j+1}}{\prod_{s=0}^j r_{t+s}^f} \geq 0. \quad (2.6)$$

This rules out the possibility that the representative household borrows to finance its consumption without limit.

The household’s utility maximization problem is characterized by: five first-order conditions, the law of motion of capital held in each period, the period budget constraint, and the non-Ponzi game condition. We are particularly interested in the optimality conditions for debt (the world risk-free asset) and capital (the risky asset) and discuss these in more detail below.

2.2 Firms

There are a large number of identical, final good producing competitive firms. Firms, which are fully owned by domestic households, produce the final goods by hiring labor and renting capital. Each firm issues a single stock which is traded domestically. This assumption about domestic ownership is made to simplify our model and also make it more theoretically tractable.

Firms use constant returns to scale production technology given by:

$$y_t = z_t h_t^\alpha k_t^{1-\alpha},$$

where the variables y_t and z_t denote output of the final good and total-factor productivity, respectively. Productivity is assumed to follow an exogenous AR(1) process defined below in Section 2.4. Since firms do not make the investment decision, their optimization problem is static. They choose k_t and h_t to maximize the current period profit, given z_t , r_t , and w_t :

$$\max_{\{k_t, h_t\}} \Pi_t = z_t h_t^\alpha k_t^{1-\alpha} - r_t k_t - w_t h_t.$$

The first-order conditions for the firms are standard and have the usual interpretation. Profits are equal to zero since we have assumed constant returns to scale technology.

2.3 The Government

The government faces a stream of public expenditures, denoted by g_t , that are exogenous, stochastic, and nonproductive. These expenditures are financed by levying the lump-sum tax Γ_t . The government's sequential budget constraint is then given by:

$$\Gamma_t = g_t, \tag{2.7}$$

for $t \geq 0$. To simplify our analysis we assume that government expenditures follow an AR(1) process. We discuss this process next.

2.4 The Driving Force

This small open economy is driven by the joint exogenous processes of productivity, the world interest rate, and government expenditures. In particular, we assume that productivity follows an independent process, while the world interest rate and government expenditures

are correlated.

The driving force is given by:

$$\begin{pmatrix} \hat{z}_t \\ \hat{r}_t^f \\ \hat{g}_t \end{pmatrix} = \begin{pmatrix} \varrho_z & 0 & 0 \\ 0 & \varrho_r & 0 \\ 0 & \varrho_{gr} & \varrho_g \end{pmatrix} \begin{pmatrix} \hat{z}_{t-1} \\ \hat{r}_{t-1}^f \\ \hat{g}_{t-1} \end{pmatrix} + \Omega \begin{pmatrix} \varepsilon_t^z \\ \varepsilon_t^{r^f} \\ \varepsilon_t^g \end{pmatrix}, \quad (2.8)$$

where \hat{z}_t and \hat{g}_t are, respectively, the Hodrick and Prescott (1997) (HP) filtered logarithm of z_t and g_t , \hat{r}_t^f is the logarithm r_t^f , and the 3×3 matrix Ω is given by

$$\Omega = \begin{pmatrix} \sigma_z & 0 & 0 \\ 0 & \sigma_r^f & 0 \\ 0 & 0 & \sigma_g \end{pmatrix}.$$

with $\sigma_z > 0$, $\sigma_r^f > 0$, and $\sigma_g > 0$. All the shocks are assumed to be independently and identically standard normal random variables.

Our specification of the exogenous driving force follows the literature. In particular, with respect to the government expenditures process, Burnside et al. (2004) and Ravn et al. (2009) have adopted similar processes. In our numerical exercise, the structural parameters ϱ_z and σ_z are calibrated. The other parameters in equation (2.8) are estimated based on the available data by applying ordinary least squares. These estimated values are given in Table 1.

2.5 Competitive Equilibrium

In equilibrium all markets are cleared and:

$$\tilde{c}_t = c_t; \tilde{h}_t = h_t. \quad (2.9)$$

The competitive equilibrium is defined in the standard form: as a sequence of real allocations $\{c_t, \tilde{c}_t, h_t, \tilde{h}_t, i_t, k_{t+1}, b_t, \Gamma_t\}_{t=0}^{\infty}$ and prices $\{r_t, w_t\}_{t=0}^{\infty}$, given $\{r_{-1}^f, d_{-1}, k_0, z_0, r_0^f, g_0\}$ and the driving force, such that households maximize utility, firms maximize profit, the government balances its budget, and all markets are cleared.

The DSGE model is solved by using perturbation methods as in, for example, Schmitt Grohé and Uribe (2004). A particularly attractive advantage of the perturbation method over other approaches is that it can easily handle a model with many state variables.

3 Data and Calibration

Our data are quarterly and we collected them from a variety of sources. To obtain equity returns, we use the MSCI market return indices data from DataStream; the country returns are computed by taking the log first differences of the market return indices. The differences between US 3-month T-bill rates and expected inflation rates are our proxy for the world risk-free rates. The expected inflation rates are given by the average inflation rates in the previous four quarters; see Uribe and Yue (2006).

The other data we use are all from the International Financial Statistics data bank. The lengths of the samples we have available vary across the different countries: the range for Chile runs from the first quarter of 1996 to the third quarter of 2007, translating into 47 observations; the sample period for Argentina covers the period between the first quarter of 1993 to the third quarter of 2007, a total of 59 observations; and the sample period for Brazil extends from the first quarter of 1991 to the third quarter of 2007, giving us 67 observations.

We transform the local-currency-denominated nominal per capita macroeconomic variables to US-Dollar-denominated real variables. All the series are deseasonalized, and the cyclical components of output, investment, and government expenditures are obtained using the HP filter.

To solve the model, we must select values for parameters which characterize the stochastic shocks, preferences, and technology. These values are chosen to keep the model roughly consistent with some empirical regularities observed in the business cycle and equity returns in the sample countries. There are two steps in the calibration process.⁴ First, for a group of parameters for each country, we determine parameter values by either setting them equal to those used or established earlier in the literature, basing them on sample means, or utilizing a steady-state optimality condition. Second, for those parameters with weak *a priori* knowledge for which we do not have strong theoretical priors, we set values to maximize the model's ability to replicate a set of business cycle and asset pricing moments.

For some parameters, we follow Mendoza (1991) and set the risk aversion coefficient, γ , to 2, the capital depreciation rate, δ , to 0.025, and the exponent of labor supply in the utility function, ω , to 1.455. Several parameters have country-specific values. We use the sample means of the corresponding data to determine the non-stochastic steady-state ratio of the trade balance to GDP, s_{tb} , the ratio of government expenditures to GDP, s_g , and the world

⁴By “calibration” we mean both assigning parameter values based upon the earlier literature and determining parameters values through a moment-matching iterative procedure. This broad interpretation is consistent with, for example, Jermann (1998).

risk-free interest rate, r^f . The labor income shares of GDP, α , are taken from the literature. In particular, we follow Michalopoulos (1969) and set its value at 0.735 for Argentina, 0.71 for Brazil, and 0.676 for Chile.

The steady-state marginal return to capital, μ_k , is calculated from the deterministic steady-state optimality condition $\mu_k = r^f - 1 + \delta$. The share of investment in value added, s_i , is calculated through the following equation:

$$s_i = \frac{i}{y} = \frac{\delta \mu_k k}{\mu_k y} = \frac{\delta s_k}{\mu_k}.$$

From the setup of the problem, the determination of the steady-state values of c and h are independent of β_1 . Thus, after we calculate the steady-state values of c and h based upon the data we have, the parameter β_1 can be calibrated from the deterministic steady-state optimality condition:

$$1 = \left(1 + c - \frac{h\omega}{\omega}\right)^{-\beta_1} r^f.$$

The calibrated value of β_1 for all countries is less than γ , which we noted above is a necessary condition to guarantee that GHH preferences have a unique limiting distribution of state variables, and that the consumption good in every period is a normal good.

Given the regression results in Table 1, we can calibrate the values of the four structural parameters ψ , ϕ , ρ_z , and σ_z . We do so by trying to match the standard deviation of output, the standard deviation of investment, the first-order autocorrelation of output, and the mean equity return through a grid search procedure. More specifically, we follow the same calibration method as used in Jermann (1998) by searching over hundreds of thousands of grid points, each defined by the quadruple formed by the particular values of these four parameters. The values of the structural parameters are listed in Table 2.

4 Results

We find that the benchmark model is able to match selected business cycle moments and the equity return.⁵ As shown in Table 3, the benchmark model can generate the standard deviation of output, the standard deviation of investment, the first-order autocorrelation of output, and the mean equity return found in the data.

The business cycle moments are in general more or less matched across countries. Com-

⁵The derivation of the equity return is shown in the Appendix.

pared to the data, our benchmark model slightly overestimates the standard deviation of output in Argentina and investment in Chile, and underestimates the first-order autocorrelation of output in Brazil.

Given the low level of the world interest rate for the sample period (around 0.8% per quarter), our model generates high equity premia in all three countries. The model matches the equity returns and equity premia in these countries very well. The best match is for Chile, for which the numerical error is on the order of 10^{-4} . Brazil is the worst case, for which the generated equity return is about 0.04 percentage points smaller than the observed sample return per quarter. Given the fact that no previous work has been able to successfully match equity premia in a small open economy, we consider these results to be particularly important. Our model can explain both business cycles and equity returns for a small open economy.

We decompose the impact of productivity, world interest rate, and government expenditure shocks on equity premia.⁶ The decomposition results are shown in Table 5. From these decompositions it is clear that productivity uncertainty is the most important factor in determining the equity premium in the long run. In Argentina 92% of the equity premium is explained by the compensation to productivity uncertainty. For Brazil this ratio is 67%, and for Chile it is 87%. This finding is in line with the literature, since in the long run productivity shocks are generally found to be the most important driving force in the economy. The second most important risk is government expenditure uncertainty. In Brazil, 30% of the equity premium is due to the compensation to uncertainty associated with this factor.

4.1 Intuition Behind the Equity Premium Result

We emphasize that this success is due to the imposition of the following three conditions: GHH utility; borrowing and lending costs; and capital adjustment costs.⁷ Omission of any one of these causes failure in generating successful results. We focus on why it is important to impose borrowing and lending costs, since the necessity of the other two factors has been discussed in the literature; see, for example, Boldrin et al. (2001), Jermann (1998), and Constantinides and Duffie (1996).

⁶In the Appendix we show how the contribution of each shock is computed.

⁷Though we have not attempted to do so, we speculate that similar results could be achieved with habit formation preferences in place of GHH utility. We decided to employ GHH preferences since the literature has established that it is suitable for analysis of small open economies; see, for example, Mendoza (1991) and Schmitt-Grohé and Uribe (2003).

Let λ_t and φ_t be the Lagrange multipliers associated with (2.2) and (2.3), respectively. The Euler equation (4.1) represents the first-order condition with respect to the debt position:

$$\left(1 + \tilde{c}_t - \frac{\tilde{h}_t^\omega}{\omega}\right)^{-\beta_1} \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} = \frac{[1 - \Psi'(d_t - \bar{d})]}{r_t^f} = \frac{1}{r_t^f \{1 + \psi [\exp(d_t - \bar{d}) - 1]\}}. \quad (4.1)$$

There are two important features about equation (4.1). First, in this small open economy $\{r_t^f\}_{t=0}^\infty$ is an exogenous process. Second, the term $1 - \Psi'(d_t - \bar{d})$ appears in the equation due to the borrowing and lending costs introduced in (2.2). Both have important implications for equity premia.

In the absence of the borrowing and lending marginal costs $[1 - \Psi'(d_t - \bar{d})]^{-1}$, the exogenous and smooth world interest rate forces the IMRS to be too smooth. Intuitively, without the imposition of these costs, the supply of foreign financial assets is quite elastic. Holding everything else constant, the effect of interest rate shocks would be absorbed by changes in the international bond holding position to keep consumption smooth. Thus, the MRS would be smooth, and the model would generate equity premia that are too low.⁸ Introduction of borrowing and lending costs makes it harder to adjust the debt position to absorb the effect of shocks, i.e., the supply of debt becomes inelastic. As a result, consumption becomes more sensitive to exogenous shocks, such that the IMRS becomes more volatile. This mechanism is analogous to the introduction of capital adjustment costs making the supply of capital inelastic; see Boldrin et al. (2001). It directly breaks down the link between the IMRS and the world risk-free interest rate, and makes the supply of international bonds less elastic.

Equation (4.1) is the key to generating a sufficiently high equity premium when moving from a closed economy to a small open economy. Without borrowing and lending costs, equation (4.1) reduces to a standard bond pricing equation and the IMRS is, as a result, forced to be too smooth. To demonstrate this point, we set $\psi = 0$ in equation (4.1) while keeping the values of other parameters unchanged. The numerical results are shown in Table 4. Note that the model-generated equity premia are all too low in this case.

By comparing the results shown in Table 4 to those in Table 3, we argue that in a small open economy it is important to impose borrowing and lending costs in order to generate sufficiently high equity returns. Further, our model does not exhibit the “risk-free rate volatility problem,” because the smooth world risk-free rates are exogenous with respect to a small open economy; see, for example, Cochrane (2006), Boldrin et al. (2001), and Jermann

⁸This is also a problem in the closed economy. Both Jermann (1998) and Boldrin et al. (2001) match the equity premia at the cost of generating an excessively high volatility in the risk-free interest rate.

(1998).

4.2 Discussion of Business Cycle Results

We report model-generated impulse-responses of aggregate consumption and output to productivity, world interest rate, and government expenditure shocks. Our interest in doing so is as follows. First, our framework provides a new mechanism to explain the sensitivity of current consumption to current income and past interest rates. To numerically explore the relationship between current consumption and current income, we require the impulse-responses of both consumption and output to productivity shocks. So, we quantitatively compute the response of output to productivity shocks even though it has been widely studied in the small open economy literature; see Mendoza (1991) and Schmitt-Grohé and Uribe (2003), among others. Our impulse-response approach to studying consumption provides, we believe, a useful complement to the two-stage regression procedure of Boldrin et al. (2001).⁹ Second, with a few exceptions such as Ravn et al. (2009), government expenditure shocks have been less studied in the DSGE literature about small open economies. Both Burnside et al. (2004) and Ravn et al. (2008), for example, consider government expenditure shocks in a closed economy, and Mendoza (1991) does not discuss the impulse-responses of key macroeconomic variables to government expenditure shocks. Accordingly, we decided to analyze the effects of government expenditure shocks.

Figure 1 plots the response of consumption and output to a 1% increase in productivity. Current consumption increases by 2.26% in Argentina, 2.03% in Brazil, and 2.55% in Chile, from its corresponding non-stochastic steady state values. Current output increases by 2.02% in Argentina, 1.95% in Brazil, and 1.87% in Chile, from its corresponding non-stochastic steady state values. Combining the non-stochastic steady-state values of consumption and output, the above results imply the following: 68% of the change of current output in Argentina is manifested as a change in current consumption; in Brazil and Chile this effect is 54% and 73%, respectively. These findings show that current consumption in our model responds significantly to a temporary contemporaneous change in output. Thus, the excessive sensitivity of consumption to income identified in Campbell and Mankiw (1989) is not a puzzle according to our representative agent optimization model.

Figure 2 plots the response of consumption to a 1% increase in the world interest rate. The fact that response functions are quite flat implies that the trade-off between current

⁹The impulse response functions directly measure the responses of variables of interest to structural shocks. Accordingly, there is no endogeneity problem to be addressed.

consumption and future consumption does not respond to a change in today's risk-free interest rate. The intuition behind this result is that, due to our use of GHH utility, the IES in consumption, compared to that associated with the CRRA preferences case, is low. This result is in line with the empirical finding in Campbell and Mankiw (1989) and the theoretical prediction in Boldrin et al. (2001). However, our results also establish that factor-market inflexibilities are not necessary conditions to explain the small response of consumption to changes in risk-free interest rates.

Figure 3 plots the response of consumption and output to a 1% increase in government expenditures. Consumption drops because of a negative wealth effect. As the government uses more resources, fewer resources are available for households. This decreases households' incomes and thus lowers consumption. We note that this result contrasts with what is frequently reported in the literature on the response of private consumption to a positive government expenditure shock in a small open economy; see, for example, Ravn et al. (2009). Output does not change initially but decreases later since the economy accumulates relatively less capital through a crowding-out effect. As government expenditures increase, the country's debt position worsens. This leads to an increase in the effective interest rate as the markup over the world risk-free rate increases due to borrowing and lending costs. As a result investment decreases, which generates subsequent reductions in output.

4.3 Sensitivity Analysis: Working-Capital Constraint

So far, output does not drop in response to a positive world interest rate shock. This is due to the fact that with GHH utility, there is no wealth effect on labor supply; see, for example, Chari et al. (2005). We are interested in modifying our model to produce an output drop following a risk-free interest rate shock and examining whether, with this extension, the model is still able to match the equity returns and other key business cycle moments found in the data. To enable the model to replicate such a decrease in output, we impose a working-capital constraint following Uribe and Yue (2006), which takes the form:

$$WK_t \geq \eta w_t h_t, \tag{4.2}$$

where the variable WK_t denotes the amount of working-capital and $\eta > 0$.

The representative firm's debt position, d_t^F , evolves as:

$$d_t^F = r_{t-1}^* d_{t-1}^F - y_t + w_t h_t + \mu_t k_t + \pi_t + WK_t - WK_{t-1}.$$

where $r_{t-1}^* = r_{t-1}^f \{1 + \psi [\exp(d_t - \bar{d}) - 1]\}$ is the effective interest rate. Defining the net liability of the representative firm as $a_t = r_t^* d_t^F - WK_t$, we can rewrite the representative firm's budget constraint as:

$$\frac{a_t}{r_t^*} = a_{t-1} - y_t + w_t h_t + \mu_t k_t + \pi_t + \left(\frac{r_t^* - 1}{r_t^*} \right) WK_t. \quad (4.3)$$

Since the representative firm is owned by the representative household, the objective function of firms is defined by:

$$\max \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \theta_t \frac{\lambda_t}{\lambda_0} \pi_t,$$

where λ_t denotes the marginal wealth utility of the representative household. The representative firm is also subject to the following non-Ponzi-game constraint:

$$\lim_{j \rightarrow \infty} \mathbb{E}_t \frac{a_{t+j}}{\prod_{s=0}^j r_{t+s}^*} \leq 0. \quad (4.4)$$

The introduction of the working-capital constraint will only change the optimality condition for labor demand.

Labor demand is determined by the following equation:

$$w_t \left[1 + \eta \left(\frac{r_t^* - 1}{r_t^*} \right) \right] = \alpha z_t k_t^{1-\alpha} h_t^{\alpha-1}. \quad (4.5)$$

Since any real-valued process $\{a_t\}_{t=0}^{\infty}$ which satisfies (4.3) and (4.4) will be optimal for the representative firm, we follow Uribe and Yue (2006) and set $a_t = 0$. Hence, only one parameter (η) in equation (4.2) needs to be calibrated. Once again following Uribe and Yue (2006), we set $\eta = 1.2$, which means that the representative firm needs to save money to be able to pay the wage bill for at least 1.2 quarters.

Table 3 reports our numerical results with the working-capital constraint. It is clear that the working-capital constraint has only a small impact on both business cycle and equity premium moments. Once we shut down the borrowing and lending costs channel, the introduction of the working-capital constraint cannot generate sufficiently high equity returns, as shown in Table 4. Indeed, the results reported in this table suggest that, in this case, the working-capital constraint has practically no effect on the model's ability in generating equity premia of the appropriate size.

5 Conclusions

The model we develop in this paper is the first in the literature to match key business cycle moments and long-run equity returns in a small open economy with production; we do so using data from Argentina, Brazil, and Chile. We obtain these results through three modifications to a standard real business cycle model: introducing borrowing and lending costs; imposing capital adjustment costs; and assuming GHH preferences. Our main finding is that the borrowing and lending cost constraint is crucial for our model to generate long-run equity returns that are sufficiently high to replicate what is observed in the data. Though it is useful in matching additional business cycle moments, the working-capital constraint we also analyze can not play the role of the borrowing and lending cost channel in terms of producing equity returns of the appropriate size. Our analysis also establishes that it is useful to consider using GHH preferences, in addition to habit formation and Epstein-Zin utility, in modeling asset-pricing behavior. In sum, we believe the model makes significant progress in addressing the equity premium puzzle for a small open economy with production.

When we decompose the contributions of productivity, world interest rate, and government expenditure shocks to the long-run equity premium, we find that productivity shocks are the most important factor behind equity premia in a small open economy. For Argentina, Brazil, and Chile, we respectively find that 92%, 67%, and 87% of the long-run equity premium is explained by the compensation to productivity uncertainty. These results are consistent with results reported in the real business cycle literature on the dominant long-run driving force role played by productivity shocks. We believe these are new results for the small open economy literature.

Our model provides a benchmark for doing policy analysis in a small open economy. Following Barro (2009), if a model can not explain the key features of asset prices, i.e., does not satisfy the “Atkeson-Phelan principle”, the welfare analysis of consumption uncertainty based on the model is less meaningful. Since our model is the first which satisfies the “Atkeson-Phelan principle” in the small open economy context with production, we arguably provide an appropriate framework for analyzing the welfare effects of macroeconomic fluctuations in such a setting. Further, since the smooth world risk-free interest rate is taken as given, our small open economy model does not exhibit the “risk-free rate volatility problem” generically present in the large open economy case.

We believe the results of this paper will prove useful to those interested in estimating endowment or production-based DSGE models with asset pricing. More specifically, we have in mind how our paper can be helpful for researchers considering Bayesian estimation

of such models; Gallant and McCulloch (2009), for example, argue that through careful use of priors Bayesian methods can overcome the problems facing classical statistical analysis of these models. Given our success in replicating business cycle moments and long-run equity returns, the calibration parameters we use can serve as priors for Bayesian estimation of asset-pricing DSGE models for the small open economies we study.

There is, however, scope for improvement in our analysis. First, we are unable to match the second moment of long-run equity returns. In this respect our work is similar to Jermann (1998), but differs from Boldrin et al. (2001); the latter can match the Sharpe ratio in the US data used in Cecchetti et al. (1993). In future work we hope to resolve this issue for the small open economy with production framework we study in this paper.

Second, the model generates some counterfactual results. In particular, consumption drops in the presence of a positive government expenditure shock, which is in contrast to what is found in the data. A possible resolution would be to extend the model by introducing relative deep habits as in Ravn et al. (2009).

Third, the model does not include investment-specific shocks, which the literature has argued are an important driving force of business cycles. Generalizing the model to allow for investment-specific shocks is an additional extension we plan to consider in later work.

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Table 1: Estimated Parameters of the Driving Force

Country	ϱ_r	ϱ_{gr}	ϱ_g	σ_r	σ_g
Argentina	0.9986	1.2008	0.8206	0.001	0.1134
Brazil	0.9834	0.3565	0.7924	0.001	0.1225
Chile	0.9882	-0.0778	0.8296	0.001	0.0349

Notes: Reported values are OLS parameter estimates of the driving force for each economy as specified in $\hat{\mathbf{s}}_t = \varrho \hat{\mathbf{s}}_{t-1} + \boldsymbol{\varepsilon}_t$, where $\hat{\mathbf{s}}_t = (\hat{z}_t, \hat{r}_t^f, \hat{g}_t)'$, \hat{z}_t and \hat{g}_t are the HP-filtered logarithm of z_t and g_t , \hat{r}_t^f is the logarithm r_t^f , z_t is total factor productivity, r_t^f is the world interest rate, g_t is government expenditures, and $\boldsymbol{\varepsilon}_t = (\varepsilon_t^z, \varepsilon_t^r, \varepsilon_t^g)'$. The parameters ϱ_r and ϱ_g represent autoregressive terms in the driving force for, respectively, the world interest rate and government expenditures, and ϱ_{gr} is the interaction term between government expenditures and world interest rates. σ_r and σ_g represent the standard errors of regression of the OLS fitted equations for the world interest rate and government expenditure processes. See equation (2.8).

Table 2: Structural Parameter Calibration

Country	ρ_z	σ_z	ϕ	ψ	γ	ω	δ	s_{tb}	s_g	α
Argentina	0.70	2.5e-3	10	4.00	2	1.455	0.1	1.00e-02	0.127	0.735
Brazil	0.50	4.0e-3	50	0.14	2	1.455	0.1	2.29e-02	0.195	0.710
Chile	0.55	3.5e-4	34	7.10	2	1.455	0.1	5.01e-02	0.115	0.676

Notes: Reported values are calibrated parameters in this study. ρ_z and σ_z are, respectively, the first-order autoregressive parameter and the standard deviation of the productivity process. ϕ and ψ are, respectively, the cost parameters for capital adjustment costs and borrowing and lending costs. γ is the coefficient of risk aversion, ω is the GHH exponent of labor supply, and δ is the capital depreciation rate parameter. s_{tb} and s_g represent, respectively, the non-stochastic steady-state ratio of the trade balance to GDP and the ratio of government expenditures to GDP. α is the labor share of national income.

Table 3: Equity Returns, Equity Premia and Selected Business Cycle Moments

Country		Business Cycle Moments			Equity Returns	Equity Premia
		σ_Y	σ_I	$\rho(Y_t, Y_{t-1})$	$\mathbb{E}_t(r_{t+1})$	$\mathbb{E}_t(r_{t+1} - r_t^f)$
Argentina	Model	18.3	22.1	83	3.98	3.16
	WK	18.4	21.8	83	4.04	3.22
	Data	17.2	22.2	81	4.00	3.18
Brazil	Model	18.3	20.5	70	5.71	4.87
	WK	18.1	19.4	69	5.64	3.81
	Data	18.3	19.9	82	5.75	4.92
Chile	Model	6.71	8.72	83	2.01	1.21
	WK	7.23	9.10	85	2.08	1.28
	Data	6.73	10.8	85	2.01	1.21

Notes: All values are in percentages. σ_Y and σ_I are, respectively, the standard deviations of output and investment, $\rho(Y_t, Y_{t-1})$ denotes the first-order autocorrelation of output, and $\mathbb{E}_t(r_{t+1})$ is the expected value of equity returns. Rows labeled as “Model” refer to our GHH utility-based model with capital adjustment and borrowing and lending costs. Rows labeled as “WK” refer to our model with the imposition of the working-capital constraint to generate an output drop, following the formulation in Uribe and Yue (2006). Rows labeled as “Data” report the unconditional sample moments.

Table 4: Equity Returns and Equity Premia without Borrowing and Lending Constraint

Country		Equity Returns	Equity Premia
		$\mathbb{E}_t(r_{t+1})$	$\mathbb{E}_t(r_{t+1} - r_t^f)$
Argentina	Model	0.98	0.16
	WK	1.00	0.18
	Data	4.00	3.18
Brazil	Model	1.14	0.31
	WK	1.13	0.30
	Data	5.75	4.92
Chile	Model	0.88	0.08
	WK	0.89	0.09
	Data	2.01	1.21

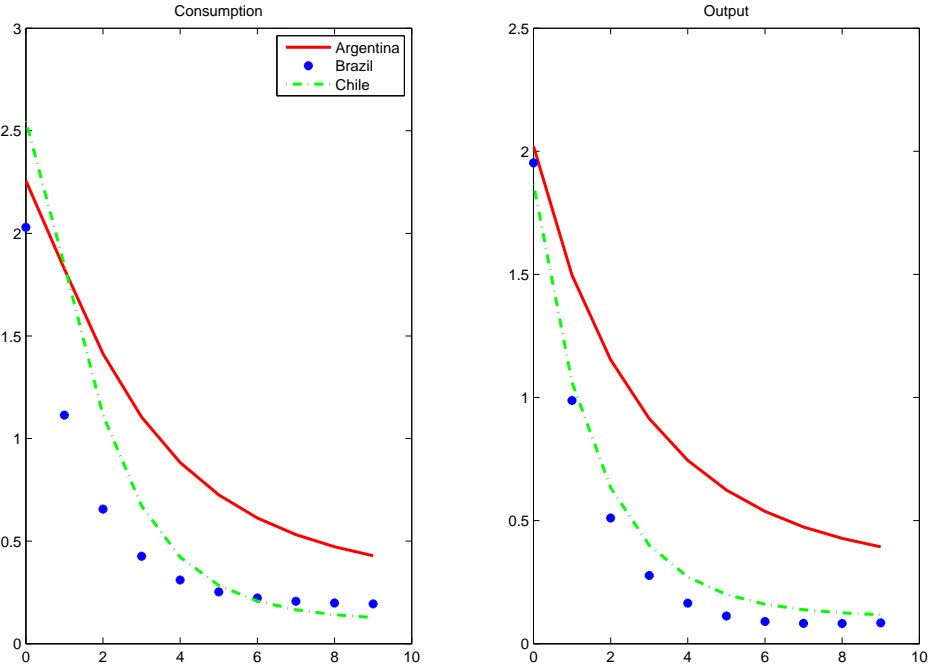
Notes: See notes to Table 3. This table demonstrates the impact of shutting down the borrowing and lending constraint. By imposing $\psi = 0$, equation (4.1) collapses to the usual RBC risk-free asset pricing equation.

Table 5: Decomposition of Equity Premia

Country	ε_t^z	$\varepsilon_t^{r^f}$	ε_t^g
Argentina	92	0	8
Brazil	67	3	29
Chile	87	8	5

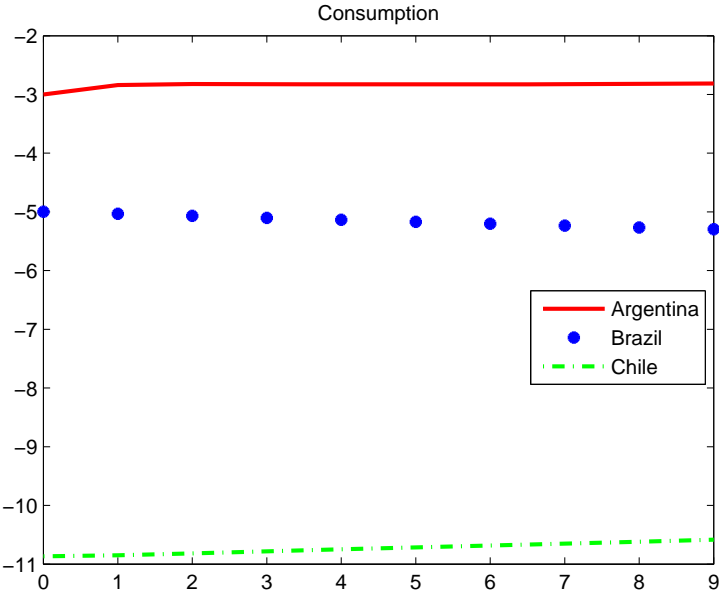
Notes: All values are in percentages. The table reports the contribution of a shock to each of the three exogenous state variables, i.e., a productivity shock (ε_t^z), a world interest rate shock ($\varepsilon_t^{r^f}$), and a government expenditures shock (ε_t^g), on the level of the equity premium in each country.

Figure 1: Response of Consumption and Output to a 1% Productivity Shock.



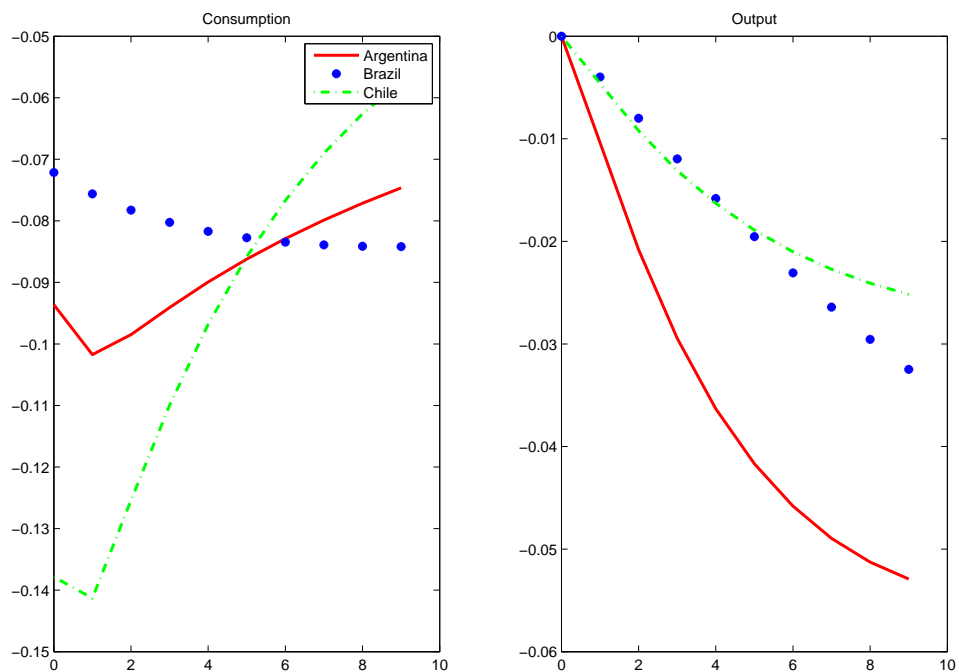
Notes: The values on the vertical axis represent deviations from the non-stochastic steady state, normalized to zero, and the values on the horizontal axis depict number of periods after initial shock. The unit of measurement for both axes is percentages. The impulse-response functions were obtained by computing the values of consumption and output after applying a 1% productivity shock to the system.

Figure 2: Response of Consumption to a 1% World Interest Rate Shock.



Notes: See notes to Table 1. The impulse-response functions were obtained by computing the values of consumption after applying a 1% world interest rate shock to the system.

Figure 3: Response of Consumption and Output to a 1% Government Expenditure Shock.



Notes: See notes to Table 1. The impulse-response functions were obtained by computing the values of consumption and aggregate output after applying a 1% government expenditures shock to the system.

6 Appendix

6.1 Asset Pricing

6.1.1 Definition

We have the following Euler equation with respect to the choice for tomorrow's capital:

$$1 = E_t \left[\frac{\left(1 + c_t - \frac{h_t^\omega}{\omega}\right)^{-\beta_1} \left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\gamma}}{\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\gamma}} \right] \frac{q_{t+1}}{p_t}, \quad (\text{A-1})$$

where:

$$p_t = 1 + \phi(k_{t+1} - k_t), \quad (\text{A-2})$$

$$q_{t+1} = 1 - \delta + \phi(k_{t+2} - k_{t+1}) + r_{t+1}. \quad (\text{A-3})$$

The equity return is defined as:

$$r_{t+1}^b = \frac{q_{t+1}}{p_t}, \quad (\text{A-4})$$

and the pricing kernel is defined as:

$$M_{t+1} = \frac{\left(1 + c_t - \frac{h_t^\omega}{\omega}\right)^{-\beta_1} \left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)^{-\gamma}}{\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\gamma}}. \quad (\text{A-5})$$

Before we derive the first-order approximation solution to the equity return and equity premium, we discuss the first-order approximation solution to our DSGE model first. There are five state variables in the model. Three of them are exogenous, z_t , r_t^f , and g_t , and two are predetermined, k_t and d_t . Let $s_t = [k_t, d_t, z_t, r_t^f, g_t]'$. Then the first-order approximation

solutions for equity returns are given by:

$$\hat{c}_t = A_c \times \hat{s}_t, \quad (\text{A-6})$$

$$\hat{h}_t = A_h \times \hat{s}_t, \quad (\text{A-7})$$

$$\hat{r}_t = A_r \times \hat{s}_t, \quad (\text{A-8})$$

$$\hat{d}_{t+1} = A_d \times \hat{s}_t, \quad (\text{A-9})$$

$$\hat{k}_{t+1} = A_k \times \hat{s}_t, \quad (\text{A-10})$$

$$\hat{s}_{t+1} = \rho \hat{s}_t + \Lambda \sigma \varepsilon_{t+1}, \quad (\text{A-11})$$

where

$$\rho = \begin{pmatrix} A_k^1 & A_k^2 & A_k^3 & A_k^4 & A_k^5 \\ A_d^1 & A_d^2 & A_d^3 & A_d^4 & A_d^5 \\ 0 & 0 & \varrho_z & 0 & 0 \\ 0 & 0 & 0 & \varrho_r & 0 \\ 0 & 0 & 0 & \varrho_{gr} & \varrho_g \end{pmatrix}, \quad e_{t+1} = \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^r \\ \varepsilon_{t+1}^g \end{pmatrix}, \quad \text{and } \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_z & 0 & 0 \\ 0 & 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 & \sigma_g \end{pmatrix},$$

$A_k = \begin{pmatrix} A_k^1 & A_k^2 & A_k^3 & A_k^4 & A_k^5 \end{pmatrix}$, and $A_d = \begin{pmatrix} A_d^1 & A_d^2 & A_d^3 & A_d^4 & A_d^5 \end{pmatrix}$. Here $\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$ and \bar{x} denotes the non-stochastic steady state of variable x . The vector A 's are functions of the steady state and structural parameters. Their values come from the numerical solution.

Once we numerically obtain (A-6)-(A-11) numerically, we can use them to derive the equity return and equity premium. That is what we show next.

6.1.2 Log-Linearizing p_t

$$\hat{p}_t = \phi k \hat{k}_{t+1} - \phi k \hat{k}_t = \phi k A_k \times \hat{s}_t - \phi k \hat{k}_t. \quad (\text{A-12})$$

6.1.3 Log-Linearizing q_{t+1}

$$\begin{aligned} \hat{q}_{t+1} &= \frac{\phi}{1 - \delta + \bar{r}} \hat{k}_{t+2} - \frac{\phi}{1 - \delta + \bar{r}} \hat{k}_{t+1} + \frac{\bar{r}}{1 - \delta + \bar{r}} \hat{r}_{t+1} \\ &= \frac{\phi}{1 - \delta + \bar{r}} (A_k \times \hat{s}_{t+1} - A_k \times \hat{s}_t) + \frac{\bar{r}}{1 - \delta + \bar{r}} A_r \times \hat{s}_{t+1}. \end{aligned} \quad (\text{A-13})$$

6.1.4 Log-Linearizing r_t

It is clear from the optimality condition that:

$$\begin{aligned}
h_t^{\omega-1} &= \alpha z_t k_t^{1-\alpha} h_t^{\alpha-1} \\
\Rightarrow h_t &= (\alpha z_t k_t^{1-\alpha})^{\frac{1}{\omega-\alpha}} \\
\Rightarrow \hat{r}_{t+1} &= \hat{z}_{t+1} + (-\alpha) \hat{k}_{t+1} + \alpha \hat{h}_{t+1} \\
&= \hat{z}_{t+1} + (-\alpha) \hat{k}_{t+1} + \frac{\alpha}{\omega-\alpha} \hat{z}_{t+1} + \frac{\alpha(1-\alpha)}{\omega-\alpha} \hat{k}_{t+1} \\
&= \frac{\omega}{\omega-\alpha} \hat{z}_{t+1} + \frac{\alpha-\omega-\omega(1-\alpha)}{\omega-\alpha} \hat{k}_{t+1}
\end{aligned}$$

6.1.5 Log-Linearizing Equity Returns

We can represent the stochastic process of equity returns as:

$$\begin{aligned}
\hat{r}_{t+1}^b &= \frac{\phi}{1-\delta+\bar{r}} (A_k \times \hat{s}_{t+1} - A_k \times \hat{s}_t) + \frac{\bar{r}}{1-\delta+\bar{r}} A_r \times \hat{s}_{t+1} \\
&\quad - (\phi k A_k \times \hat{s}_t - \phi k \hat{k}_t).
\end{aligned} \tag{A-14}$$

6.1.6 Log-Linearizing the Stochastic Discount Factor

From (A-5), we can write the pricing kernel as:

$$\log(M_{t+1}) = -\beta_1 \log\left(1 + c_t - \frac{h_t^\omega}{\omega}\right) - \gamma \log\left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right) + \gamma \log\left(c_t - \frac{h_t^\omega}{\omega}\right)$$

So, we have:

$$\begin{aligned}
m_{t+1} &= \log(M_{t+1}) = -A_t^1 - A_t^2 + A_t^3 \\
\bar{m} &= -\bar{A}^1 - \bar{A}^2 + \bar{A}^3
\end{aligned} \tag{A-15}$$

The approximation of A_t^1 is given by:

$$A_t^1 = \bar{A}^1 + \kappa_c^1 \hat{c}_t + \kappa_h^1 \hat{h}_t, \tag{A-16}$$

where:

$$\begin{aligned}\kappa_c^1 &= \frac{\partial A_t^1}{\partial \log(c_t)} \Big|_{\text{Nonstochastic Steady State}} = \frac{\beta_1 \bar{c}}{\left(1 + \bar{c} - \frac{\bar{h}^\omega}{\omega}\right)} \\ \kappa_h^1 &= \frac{\partial A_t^1}{\partial \log(h_t)} \Big|_{\text{Nonstochastic Steady State}} = \frac{-\beta_1 \bar{h}^\omega}{\left(1 + \bar{c} - \frac{\bar{h}^\omega}{\omega}\right)}\end{aligned}$$

The approximation of A_t^2 is given by:

$$A_t^2 = \bar{A}^2 + \kappa_c^2 \hat{c}_{t+1} + \kappa_h^2 \hat{h}_{t+1}, \quad (\text{A-17})$$

where

$$\begin{aligned}\kappa_c^2 &= \frac{\gamma c_{t+1}}{\left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)} \Big|_{\text{Nonstochastic Steady State}} = \frac{\gamma \bar{c}}{\left(\bar{c} - \frac{\bar{h}^\omega}{\omega}\right)} \\ \kappa_h^2 &= \frac{-\gamma h_{t+1}^\omega}{\left(c_{t+1} - \frac{h_{t+1}^\omega}{\omega}\right)} \Big|_{\text{Nonstochastic Steady State}} = \frac{-\gamma \bar{h}^\omega}{\left(\bar{c} - \frac{\bar{h}^\omega}{\omega}\right)}\end{aligned}$$

The approximation of A_t^3 is given by:

$$A_t^3 = \bar{A}^3 + \kappa_c^3 \hat{c}_t + \kappa_h^3 \hat{h}_t, \quad (\text{A-18})$$

where

$$\begin{aligned}\kappa_c^3 &= \frac{\gamma c_t}{\left(c_t - \frac{h_t^\omega}{\omega}\right)} \Big|_{\text{Nonstochastic Steady State}} = \frac{\gamma \bar{c}}{\left(\bar{c} - \frac{\bar{h}^\omega}{\omega}\right)} \\ \kappa_h^3 &= \frac{-\gamma h_t^\omega}{\left(c_t - \frac{h_t^\omega}{\omega}\right)} \Big|_{\text{Nonstochastic Steady State}} = \frac{-\gamma \bar{h}^\omega}{\left(\bar{c} - \frac{\bar{h}^\omega}{\omega}\right)}\end{aligned}$$

Combining (A-15)-(A-18), we have:

$$\begin{aligned}m_{t+1} &= A_t^3 - A_t^1 - A_t^2 \\ &= \left(\bar{A}^3 + \kappa_c^3 \hat{c}_t + \kappa_h^3 \hat{h}_t\right) - \left(\bar{A}^1 + \kappa_c^1 \hat{c}_t + \kappa_h^1 \hat{h}_t\right) - \left(\bar{A}^2 + \kappa_c^2 \hat{c}_{t+1} + \kappa_h^2 \hat{h}_{t+1}\right) \\ &= \left(\bar{A}^3 - \bar{A}^1 - \bar{A}^2\right) + \left(\kappa_c^3 - \kappa_c^1\right) \hat{c}_t + \left(\kappa_h^3 - \kappa_h^1\right) \hat{h}_t - \left(\kappa_c^2 \hat{c}_{t+1} + \kappa_h^2 \hat{h}_{t+1}\right)\end{aligned} \quad (\text{A-19})$$

Note that in the non-stochastic steady state:

$$\bar{A}^3 = \bar{A}^2.$$

We can simplify (A-19) as:

$$\begin{aligned} m_{t+1} &= -\bar{A}^1 + (\kappa_c^3 - \kappa_c^1) \hat{c}_t + (\kappa_h^3 - \kappa_h^1) \hat{h}_t - (\kappa_c^2 \hat{c}_{t+1} + \kappa_h^2 \hat{h}_{t+1}) \\ &= -\bar{A}^1 + \chi_1 \hat{c}_t + \chi_2 \hat{h}_t + \chi_3 \hat{c}_{t+1} + \chi_4 \hat{h}_{t+1} \end{aligned} \quad (\text{A-20})$$

Plugging (A-6), (A-7), and the solutions to \hat{r}_{t+1} and \hat{k}_{t+1} into (A-20), we get the following:

$$\begin{aligned} m_{t+1} + \bar{A}^1 = \hat{m}_{t+1} &= \chi_1 A_c \times \hat{s}_t + \chi_2 A_h \times \hat{s}_t \\ &\quad + \chi_3 A_d \times \hat{s}_{t+1} + \chi_4 A_k \times \hat{s}_{t+1}. \\ &= (\chi_1 A_c + \chi_2 A_h) \hat{s}_t + (\chi_3 A_d + \chi_4 A_k) \hat{s}_{t+1} \\ &= \Delta_1 \hat{s}_t + \Delta_4 \hat{s}_{t+1}, \end{aligned} \quad (\text{A-21})$$

where $\Delta_1 = (\chi_1 A_c + \chi_2 A_h)$ and $\Delta_4 = (\chi_3 A_d + \chi_4 A_k)$. Using (A-11) in (A-21), we get:

$$\hat{m}_{t+1} = \Delta_1 \hat{s}_t + \Delta_4 (\rho \hat{s}_t + \Lambda \sigma e_{t+1}). \quad (\text{A-22})$$

6.1.7 Equity Premium and Equity Returns

The equity premium is given by:

$$\mathbb{E}_t r_{t+1}^b - r_t^f = \bar{r}^f (\mathbb{E}_t \hat{r}_{t+1}^b - \hat{r}_t^f),$$

where $\bar{r}^f = \bar{r}^b$. Since:

$$\begin{aligned} \hat{r}_{t+1}^b &= \frac{\phi}{1 - \delta + \bar{r}} (A_k \times \hat{s}_{t+1} - A_k \times \hat{s}_t) + \frac{\bar{r}}{1 - \delta + \bar{r}} A_r \times \hat{s}_{t+1} \\ &\quad - (\phi k A_k \times \hat{s}_t - \phi k \hat{k}_t), \\ \hat{m}_{t+1} &= \Delta_1 \hat{s}_t + \Delta_4 (\rho \hat{s}_t + \Lambda e_{t+1}), \end{aligned}$$

we get the following:

$$\begin{aligned} \hat{r}_{t+1}^b &= \left(\frac{\phi A_k}{1 - \delta + \bar{r}} + \frac{\bar{r} A_r}{1 - \delta + \bar{r}} \right) \hat{s}_{t+1} = \left(\frac{\phi A_k}{1 - \delta + \bar{r}} + \frac{\bar{r} A_r}{1 - \delta + \bar{r}} \right) \Lambda e_{t+1} \\ \hat{m}_{t+1} &= \Delta_4 (\rho \hat{s}_t + \Lambda e_{t+1}) = \Delta_4 \Lambda e_{t+1} \end{aligned}$$

So, the equity premium is given by:

$$\mathbb{E}_t r_{t+1}^b - r_t^f = -\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1} \right\},$$

and the equity return is given by:

$$\mathbb{E}_t r_{t+1}^b = r_t^f - \bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1} \right\},$$

where:

$$e_{t+1} = \left(0 \ 0 \ \varepsilon_{t+1}^z \ \varepsilon_{t+1}^r \ \varepsilon_{t+1}^g \right)'$$

Define:

$$\begin{aligned} e_{t+1}^z &= \left(0 \ 0 \ \varepsilon_{t+1}^z \ 0 \ 0 \right), \\ e_{t+1}^r &= \left(0 \ 0 \ 0 \ \varepsilon_{t+1}^r \ 0 \right), \\ e_{t+1}^g &= \left(0 \ 0 \ 0 \ 0 \ \varepsilon_{t+1}^g \right). \end{aligned}$$

Then, the contribution of each shock is given by:

$$\begin{aligned} \text{contribution from } z \text{ shock} &= \frac{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}^z, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1}^z \right\}}{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1} \right\}} \times 100 \\ \text{contribution from } r^f \text{ shock} &= \frac{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}^r, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1}^r \right\}}{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1} \right\}} \times 100 \\ \text{contribution from } g \text{ shock} &= \frac{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}^g, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1}^g \right\}}{-\bar{r}^f \times \text{COV}_t \left\{ \Delta_4 \Lambda e_{t+1}, \left(\frac{\phi A_k}{1-\delta+\bar{r}} + \frac{\bar{r} A_r}{1-\delta+\bar{r}} \right) \Lambda e_{t+1} \right\}} \times 100 \end{aligned}$$

6.2 Intertemporal Elasticity of Substitution for GHH Utility:

The Euler equation with respect to the bond holding position without borrowing and lending is given by:

$$\lambda_t = \beta r_t^f \lambda_{t+1}.$$

Thus, we have:

$$\frac{\partial \ln \left(\frac{U_{c_{t+1}}}{U_{c_t}} \right)}{\partial \ln r_t^f} = -1.$$

Since we are interested in the intertemporal elasticity of substitution (IES), we need to calculate the value of $\partial \ln \left(\frac{C_{t+1}}{C_t} \right) / \partial \ln r_t^f$. To obtain what we want, we compute a first-order Taylor approximation of $\ln U_{c_t}$ and $\ln U_{c_{t+1}}$:

$$\begin{aligned} \ln U_{c_{t+1}} &= \ln(U_c) + \left[\frac{\partial \ln U_{c_{t+1}}}{\partial \ln C_{t+1}} \Big|_{C_{t+1}=\bar{C}} \right] [\ln C_{t+1} - \bar{C}], \\ \ln U_{c_t} &= \ln(U_c) + \left[\frac{\partial \ln U_{c_t}}{\partial \ln C_t} \Big|_{C_t=\bar{C}} \right] [\ln C_t - \bar{C}]. \end{aligned}$$

It is easy to see that:

$$\begin{aligned} \ln \left(\frac{U_{c_{t+1}}}{U_{c_t}} \right) &= \frac{\partial \ln U_c}{\partial \ln C} \ln \left(\frac{C_{t+1}}{C_t} \right), \\ \text{IES} &= \frac{-1}{\frac{\partial \ln U_c}{\partial \ln C} \Big|_{C=\bar{C}}}. \end{aligned}$$

Next, consider the following expressions:

$$\begin{aligned} C - \frac{h^\omega}{\omega} &= (r^f)^{1/\beta_1} - 1, \\ \ln U_c &= -\gamma \ln \left(C - \frac{h^\omega}{\omega} \right). \end{aligned}$$

At the steady state we know that:

$$\frac{\partial \ln U_c}{\partial \ln C} = \frac{\partial \ln U_c}{\partial C} \cdot \frac{\partial C}{\partial \ln C} = \frac{-\gamma C}{C - \frac{h^\omega}{\omega}}.$$

So, gathering terms and plugging in, we have:

$$\text{IES} = \frac{-1}{\frac{\partial \ln U_c}{\partial \ln C}} = \frac{C - \frac{h^\omega}{\omega}}{\gamma C} < \frac{1}{\gamma}.$$