

# Welfare Effect of Interest Rate Volatility

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## Abstract

This paper shows that a positive relationship between volatility of interest rate shocks ( $R$ -shocks) and the ex-ante welfare of risk averse households holds with three widely used preference representations and with a wide range of structural parameter values in a 2-period model. With a log utility function, the volatility of  $R$ -shocks is welfare deteriorating when the representative household is a net lender who saves a lot. It is welfare enhancing if the representative household is a net borrower. Volatility may be detrimental to the borrower if an exogenous borrowing limit binds with a sufficiently high probability. With a general constant relative risk aversion (CRRA) utility function, there is a threshold value of the coefficient of risk aversion for any given endowment flow. When the true value is above this threshold, volatility reduces welfare; and vice versa. With a ‘habit formation’ utility, there is a threshold value of the habit formation coefficient for any given endowment flow. When the true value is above this threshold, volatility reduces welfare; and vice versa. These results have implications for the modeling of small open economies facing “sudden stops” in financing.

Keywords: Welfare; Uncertainty; Interest rate shocks; Binding borrowing limit.

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# 1 Introduction

Interest rate shocks are regarded as one important factor that drives the business cycles of emerging economies. Neumeyer and Perri (2005) finds that stabilizing international interest rates would lower Argentina output volatility by 3%; while Uribe and Yue (2006) shows that US interest rate shocks explain about 20% of movements in aggregate activity in emerging economies. To an individual household, interest rate shocks identified as monetary policy shocks are also an important factor in driving the business cycles, see Christiano et al. (2005). However, the (ex-ante) welfare implications of the volatility of interest rates for the representative household in a small open economy, or for an individual household in a closed economy, is less straightforward. Accepting conventional wisdom, since households are typically assumed to be risk averse in consumption, volatile interest rates necessarily imply volatile consumption and hence lower utility, compared to the corresponding steady state counterparts.

This paper shows that, on the contrary, volatile interest rates ( $R$ -shocks) alone, do not necessarily imply lower expected utility. In particular, a change in the volatility of interest rates has both intertemporal substitution and income effects on ex-ante welfare, when the income flow is fixed and known. Through the substitution effect, volatility brings down expected utility, because it increases the uncertainty about the substitution between today's consumption and tomorrow's consumption. However, through the income effect, the volatility of interest rates pushes up expected utility. This is because permanent income is inversely correlated with the interest rate, while the indirect utility is concave in permanent income; thus the indirect utility is convex in the  $R$ -shocks. The net welfare effect of uncertainty of interest rates depends on which effect dominates, which in turn depends on the net asset position of the agent. This is in line with the literature. Mendoza (2002) shows that the welfare benefit is increasing in the net international debt position.

The income effects are different across households with different borrowing and lending net asset positions. Hence the volatility of  $R$ -shocks has an asymmetric net welfare effect on borrowers and lenders. For lenders who save a lot, the substitution effect is likely to dominate, thus volatile interest rates are likely detrimental. While for borrowers, the income effect dominates, and volatile interest rates mean an increase in utility. Hence, if volatile interest rates are to work against a risk averse household, one channel

might be through an external borrowing limit that binds with a sufficiently high probability, lowering expected utility when borrowing is optimal. Another channel leading volatile interest rates to work against a small open economy, may be that the economy is filled with lenders who save a lot. The smaller oil exporting countries can be regarded as such lenders. For example, according to International Monetary Fund, Kuwait, from 1993 to 2007, has had a current account balance to GDP ratio averaging above 20%.

The positive relationship between the volatility of  $R$ -shocks and ex-ante welfare holds not only with the standard log utility function, but also with general constant relative risk aversion (CRRA) utility, and with a general version of habit formation utility, see Boldrin et al. (2001), Jermann (1998). The positive relationship holds over a wide range of structural parameter values. This raises a question about the optimality of countercyclical policy, which is almost universally recommended.

This paper contributes to the literature by studying the welfare impact of interest rate shocks on a representative household, with known income stream, in a small open economy. Many other shocks have been studied in the similar fashion. Cho and Cooley (2005) shows that the standard concave utility function can be a convex function of productivity in a closed economy. When the business cycles are driven by monetary shocks, both Obstfeld and Rogoff (2000), and Bacchetta and Wincoop (2000) show high exchange rate volatility may lead to higher welfare of the risk averse households. Within a partial equilibrium framework, Oi (1961) shows a positive relationship between utility and the variance of price shocks.

The rest of the paper is organized as follows. Section 2 sets up the model and shows why volatile  $R$ -shocks are welfare-improving with standard time-separable log utility. Sections 3 and 4 extend the discussion to general time-separable CRRA preferences and to non-separable ‘habit formation’ preferences. Section 5 concludes.

## 2 The Model

Suppose a representative household without any initial saving lives in a small open economy for two periods, labeled 1 and 2. The representative household receives utility from its consumption in both periods. Its preferences are represented by a standard time-separable log utility function. The endowment flow is known and exogenously given as:  $y_1$  in Period 1, and  $y_2$  in the Period 2. The borrowing and lending interest rate

prevailing in the (international) capital market is  $R = (1 + r)$ .  $R$  is a random variable the exact value of which is known in Period 1 when the borrowing/lending decision must be made. We explore the impact of its variability on the ex-ante welfare (the expected utility) of this household (or small open economy).

Since  $R$  is known in Period 1, the representative household solves a perfect foresight optimization problem. Formally, it chooses consumption and borrowing (or saving, depending on the endowment flow) to maximize its utility:

$$\max_{\{c_1, c_2, d\}} U = \log(c_1) + \log(c_2)$$

s.t.

$$c_1 = y_1 + d, \tag{1}$$

$$c_2 = y_2 - Rd, \tag{2}$$

where the variable  $c_1$  denotes consumption in Period 1; the variable  $c_2$  denotes consumption in Period 2; and the variable  $d$  denotes the household's borrowing (or saving).<sup>1</sup> If  $d > 0$ , it means that the representative household is a borrower in Period 1; if  $d < 0$ , it means that the representative household is a lender in Period 1. The usual subjective discount factor is assumed to be  $\beta = 1$  to simplify the analytical solution. The budget constraints are given by Eqs.(1) and (2) for periods 1 and 2, respectively. Equation (1) says that the household will consume up to its endowment plus the amount that it decides to borrow from the rest of the world. Equation (2) shows that the household will consume whatever is left after paying back (collecting) its debt (loan) plus interest ( $R = 1 + r$ ).

It is straightforward to derive the solution to this optimization problem as:

$$c_1^* = \frac{y_1 + y_2/R}{2} = \frac{w(R)}{2}, \tag{3}$$

$$c_2^* = \frac{y_1 + y_2/R}{2/R} = \frac{w(R)}{2/R}, \tag{4}$$

$$d^* = \frac{y_2/R - y_1}{2} = \frac{w(R)}{2} - y_1, \tag{5}$$

$$U^* = v(R) = 2 \log(w(R)/2) + \log R = 2 \left[ \log \left( y_1 + \frac{y_2}{R} \right) - \log 2 \right] + \log R. \tag{6}$$

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<sup>1</sup>A choice of any one of  $\{c_1, c_2, d\}$  fixes the other two in this 2-period setting.

where  $v(R)$  is the indirect utility function and the function  $w(R)$  denotes the discounted present value of endowed, or permanent, income:  $w(R) \equiv y_1 + y_2/R$ . From this definition, it is clear that volatile  $R$ -shocks also make permanent income more volatile as long as the endowment tomorrow is nonzero; again, we are interested in whether this is welfare-improving or welfare deteriorating.

From Eq. (6), it is clear that  $R$  will affect utility through two channels which we call (i) a permanent income effect, and (ii) a substitution effect. The first (bracketed) term shows the impact of  $R$  through income,  $w(R) = y_1 + \frac{y_2}{R}$ , on welfare. The term,  $-\log(2)$ , means that both consumption today and tomorrow will increase by 0.5% if the permanent income increases by 1%, while multiplication by 2 adds up the total permanent income effect on both today's and tomorrow's consumption. The final term,  $\log(R)$ , summarizes the substitution effect of  $R$ , which is clear if we divide Eq. (3) by Eq. (4) and rearrange:

$$MRS_{12}(c_1^*, c_2^*) \equiv \frac{1/c_1^*}{1/c_2^*} = R.$$

The above Euler equation reflects the standard optimality condition that the intertemporal marginal rate of substitution in equilibrium is determined by  $R$ .

Since  $R$  is unknown ex-ante, we take the expectation on both sides of Eq. (6) to obtain the following equation:

$$\mathbb{E}_0 \{v(R, w)\} = \mathbb{E}_0 \left\{ 2 \left[ \log \left( y_1 + \frac{y_2}{R} \right) - \log 2 \right] \right\} + \mathbb{E}_0 \{ \log R \}. \quad (7)$$

where  $\mathbb{E}_0$  denotes the ex-ante expectation operator. Equation (7) connects the expected utility in the equilibrium,  $E_0 U^*$ , to the volatility of  $R$ -shocks, clearly displaying how the volatility of  $R$ -shocks will affect the expected utility through both channels.

First, the ‘‘permanent income effect’’ is summarized by the first term on the right hand side,  $\mathbb{E}_0 \{ 2 [\log (y_1 + \frac{y_2}{R}) - \log 2] \}$ . It is clear from this term that this income effect of uncertainty is positive: the present discounted value of endowments is inversely correlated with  $R$ , while the indirect utility is concave in that value. As a result, this first component of the indirect utility is convex in  $R$ , which implies that utility is increasing with volatility of  $R$ :

$$\frac{\partial^2 \{ 2 [\log (w(R)) - \log 2] \}}{\partial R^2} = \frac{4}{R^2} \frac{y_2}{Ry_1 + y_2} - \frac{2}{R^2} \frac{y_2^2}{(Ry_1 + y_2)^2} = \frac{2y_2^2 + 4Ry_1y_2}{R^4 (w(R))^2} > 0.$$

Second, the volatility of  $R$  also affects ex-ante welfare through the substitution effect. From Eq. (7), this substitution effect is represented by the second term on the right hand

side,  $\mathbb{E}_0 \{\log R\}$ . It is clear that this term is concave in  $R$ , i.e., the substitution effect of volatility is negative: when  $R$  becomes more volatile, the intertemporal substitution in consumption becomes more volatile, thus lowering the expected utility of households.

So we see that these two effects work against each other. Whether utility is increasing or decreasing in the volatility of  $R$  depends on which effect is going to dominate. To analyze how the volatility of  $R$ -shocks influences welfare, it is convenient to disentangle the case in which the representative household is a net borrower from the case when the representative household is a net lender. In the first case, we show that it is likely that the income effect dominates; while in the second case, we show it is likely that the substitution effect will dominate.

To facilitate the discussion, we assume that  $R$  is uniformly distributed over  $[R^*(1 - \sigma), R^*(1 + \sigma)]$ . The variable  $R^*$  denotes the expected value (the non-stochastic steady state of the  $R$ -process) and  $\sigma$  is a parameter determining the volatility of  $R$ -shocks. One restriction we need to put is  $R^*(1 - \sigma) \geq 1$  because the real interest rate,  $r$ , should not be less than 0. Thus, the expectation of  $R$  is given by:

$$\mathbb{E}R = \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \frac{R}{2\sigma R^*} dR = R^*.$$

It is clear that, in a fully dynamic model, the mean of  $R$  is preserved: the non-stochastic steady state is equal to its stochastic steady state. The variance of  $R$  is given by

$$\text{VAR}(R) = \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \frac{(R - R^*)^2}{2\sigma R^*} dR = \frac{2R^{*2}\sigma^2}{3}. \quad (8)$$

Equation (8) confirms that the parameter  $\sigma$  determines the variance of  $R$  for any given  $R^*$ . We now turn to two extreme cases that sharply display these “effects,” beginning with the case where the agent must save for the future.

## 2.1 Case 1: No endowment tomorrow

The simplest case for the representative household to be a net lender is to assume  $y_1 > 0$  and  $y_2 = 0$ ; for this example let  $y_1 = 2$ . With this assumption, the household does not have any endowment for consumption in Period 2. To smooth consumption, the household has to save, so it is a net lender. This can be seen by checking Eq. (5):

$$d^* = -\frac{y_1}{2} = -1 < 0. \quad (9)$$

Then, Eq. (7) becomes

$$\mathbb{E}_0 U^* = \mathbb{E}_0 \log R. \quad (10)$$

From Eq. (10), we see that the volatility of  $R$ -shocks will unambiguously decrease the utility of the household because  $\log R$  is a concave function of  $R$ . The reason is that, in this case, there is no income effect as there is no volatility of permanent income [ $w(R) = y_1$ ], but just the substitution effect from the volatility of the intertemporal marginal rate of substitution.

The result here raises an issue with respect to the use of the log-normal distribution, e.g.,  $\log R \sim \text{NIID}(0, \sigma^2)$ , to study the welfare effect of the uncertainty. With this distribution, we obtain:

$$\mathbb{E}_0 U^* = \mathbb{E}_0 \log R = 0.$$

In this case,  $\mathbb{E}_0 R = \exp(\sigma^2/2)$ , which is increasing with  $\sigma^2$ . The above result shows if the mean of  $R$  is not preserved, a misleading conclusion may arise: that the volatility of  $R$ -shocks will not have a deteriorating impact on the household's welfare.

## 2.2 Case 2: No endowment today

Similarly, we can assume  $y_1 = 0$  and  $y_2 > 0$ ; say,  $y_2 = 2$ . In this case, the household must be a borrower, as can be seen from

$$d^* = \frac{y_2}{2R} = 1/R > 0. \quad (11)$$

Equation (7) then becomes

$$\mathbb{E}_0 U^* = -\mathbb{E}_0 \log R. \quad (12)$$

From Eq. (12), it is clear that the volatility of  $R$ -shocks will unambiguously increase the utility of the household.<sup>2</sup> Mathematically, it is because the expected utility is a convex function of  $R$ , see Eq. (12). Intuitively, in this case, the income effect, which is summarized in the term,  $-2 \log(R)$ , is very strong; it dominates the substitution effect of uncertainty so that volatility enhances the expected utility.

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<sup>2</sup>Again, there will be no impact if  $\log R \sim \text{NIID}(0, \sigma^2)$ .



## 2.3 Scenarios in between

We now generalize the examples by studying the endowment flows between these two extreme cases. We assume the following:  $y_1 = 2x_1$  and  $y_2 = 2x_2$ , where  $(x_1, x_2) \in [0, 1]^2 \setminus \{(0, 0)\}$ . When  $x_1 = 1$  and  $x_2 = 0$ , this is case 1. When  $x_1 = 0$  and  $x_2 = 1$ , it is case 2. Thus, Eq. (7) becomes

$$\begin{aligned} \mathbb{E}_0 U^* &= \mathbb{E}_0 2 \log \left( x_1 + \frac{x_2}{R} \right) + \mathbb{E}_0 \log R = \mathbb{E}_0 2 \log \left( \frac{w(R)}{2} \right) + \mathbb{E}_0 \log R \\ &= \frac{1}{2R^*\sigma} \int_{R^{*(1-\sigma)}}^{R^{*(1+\sigma)}} [2 \log (x_1 R + x_2) - \log R] dR. \end{aligned} \quad (13)$$

This formulation allows us to investigate how the endowment flow affects the impact of interest rate volatility ( $R$ -shocks) on the consumer's welfare. Of particular interest is determining, for each  $x_1$ , the threshold value of  $x_2$  above which  $R$ -volatility is welfare-improving, and below which  $R$ -volatility reduces the agent's welfare (expected utility). The existence of such a threshold is indicated by the results above: when there is no endowment today, volatility is welfare-improving, while when there is no endowment tomorrow, volatility is welfare-deteriorating.

To find such a threshold value, we use the simulation method instead of analytic derivation.<sup>3</sup> For each income flow,  $(x_1, x_2)$ , we compute optimal expected utility,  $U^*$ , at each standard deviation level between 0.1% and 4% in order to find the income allocation at which  $U^*$  switches from increasing to decreasing in  $R$ -variance. Our simulation steps are as follows:

1. We set  $R^* = 1.065$ , a mean interest rate faced by a typical small open economy, see Mendoza and Smith (2005), Mendoza and Uribe (2000). The range of  $\sigma$  is set at (0.1%, 4%). This range includes the estimate of Neumeyer and Perri (2005), 1.08%. The grid width for both  $x_1$  and  $x_2$  is set at 0.001.
2. For each  $(x_1, x_2)$ , we calculate the value of utility defined in Eq. (13) for each distribution of  $R$  (i.e., for each grid point of  $\sigma$ ).<sup>4</sup> As a result, we can plot the

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<sup>3</sup>To do the simulation exercise, we use the analytical expression of Eq. (13), which is given in the Appendix in Section 7.1. Simulations here, and below are implemented in MatLab (R2008a).

<sup>4</sup>The expectation of  $R$  is fixed so that its distribution is mainly described by its variance, whose value is monotonically determined by  $\sigma$ , as shown in Eq. (8).

calculated expected utility against  $\sigma$ . With our setup, the numerical results show that the plotted curve is either upward sloping or downward sloping over the range of  $\sigma$ . If the curve for one particular  $(x_1, x_2)$  is upward sloping, we say that uncertainty is welfare-improving for this particular  $(x_1, x_2)$ ; and vice versa.

3. The threshold value of  $x_2$  is found by searching over the grid points. In particular, we fix  $x_1$ , then search over  $x_2$  until the relationship between welfare and the volatility changes from positive to negative.<sup>5</sup>

Figure 1 plots the threshold value of  $x_2$  against  $x_1$ . If the point  $(x_1, x_2)$  is above the threshold value line, the representative household will receive higher utility from more volatile  $R$ -shocks. The positive relationship is represented by the “+” sign. If below, the representative household will obtain lower utility when  $R$ -shocks become more volatile. The negative relationship is represented by the “-” sign. The two extreme cases are also indicated, illustrating how they are generalized by this analysis.

It is clear that only if the household saves a lot (below the threshold value line) because it has so little endowment tomorrow, will the volatility of  $R$ -shocks decrease its expected utility. To clarify that point, Figure 1 also plots the borrowing and lending line. If the point  $(x_1, x_2)$  is above the borrowing and lending line, the representative household is going to be a borrower; while if below, the household is going to be a lender. For any point  $(x_1, x_2)$  above the threshold value line and below the borrowing and lending line, the household is a lender yet its utility is enhanced by the volatility of  $R$ -shocks. So, it is not necessary to be a borrower to enjoy the increased uncertainty. Hence the positive relationship is more prevalent than has been apparently believed. Our result is, however, not out of line with part of the literature. Mendoza (2002), for example, shows that the welfare benefit is increasing in the net international debt position.

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<sup>5</sup>During the search, occasionally, the relationship is not monotonic for one  $x_2$  for the given  $x_1$ . With this particular pair of  $(x_1, x_2)$ : the relationship is positive (negative) for some range of  $\sigma$ ; while for some other range of  $\sigma$ , it becomes negative (positive). However, this is not a serious issue for our problem. In our simulation exercise, if we decrease that particular  $x_2$  by one grid point, or increase that particular  $x_2$  by one grid point, the relationship becomes monotonic again. So, the occurrence of non-monotonic observation may cause some disagreement on the exact value of the threshold  $x_2$ , but this disagreement will cause at most two grid widths difference (0.002) among the threshold values pinned down.

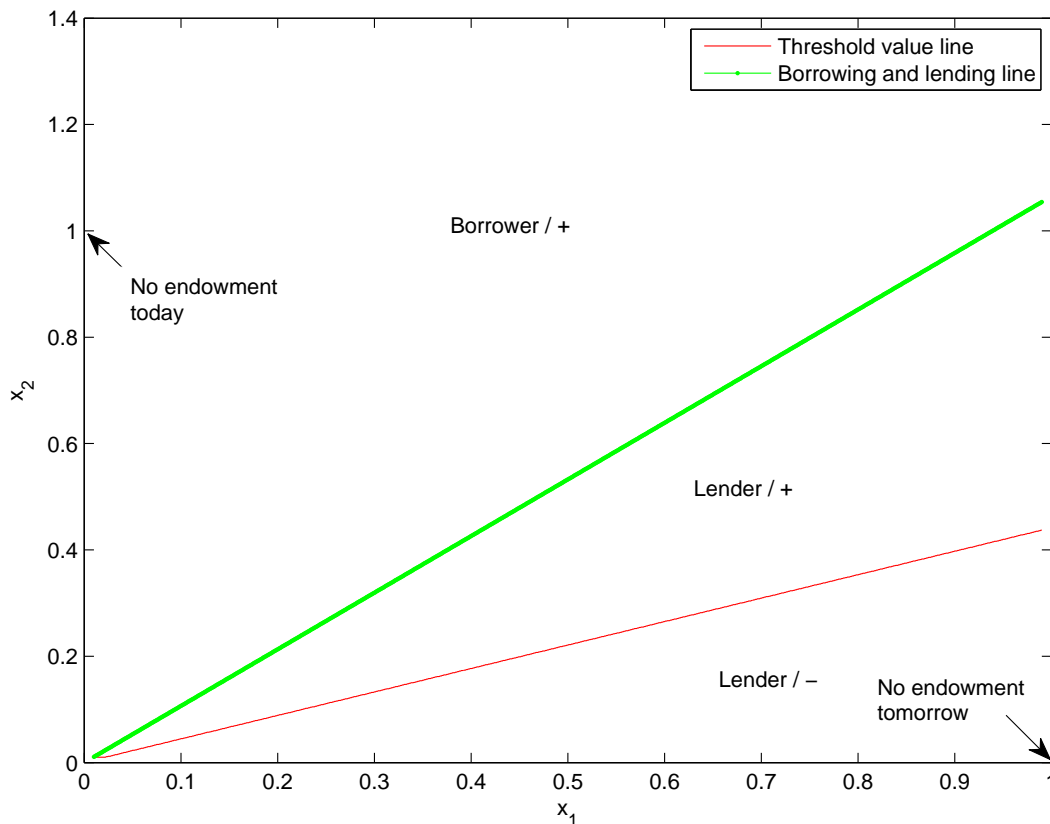


Figure 1: Threshold value line and borrowing and lending line

## 2.4 Possible reasons why volatility is welfare deteriorating?

The conventional belief about welfare effect of  $R$ -shocks is as follows: (1) In the case of an individual household, highly volatile interest rates make consumption volatile, thus reducing the welfare of the household. (2) In the case of a representative household in a small open economy, highly volatile external shocks are believed to make this small open economy less stable, thus implying lower welfare. Our story about the welfare-improving impact of the volatility of  $R$ -shocks thus seems counter-intuitive. To reconcile this theoretical prediction and conventional beliefs, especially in terms of a small open economy, we explore two possible mechanisms that may lead to welfare deterioration

from increased  $R$ -volatility: a ‘distribution effect’, and a binding borrowing limit.<sup>6</sup>

### 2.4.1 Distribution effect

The distribution effect arises from relative shares of savers and borrowers in the small open economy. Suppose in our base model that there is a continuum of households in this economy. They are indexed by  $j$ , where  $j \in [0, 1]$ . Further suppose that there is a cutoff point,  $\kappa$ , such that people in the range of  $[0, \kappa]$  must borrow, i.e., have an endowment flow of  $y_1^j = 0$  and  $y_2^j = 2$ ; while people in the rest  $(\kappa, 1]$  must save, i.e., have an endowment flow of  $y_1^j = 2$  and  $y_2^j = 0$ . In addition, we assume that the social welfare is strictly utilitarian and is given by the sum of each individual’s welfare. So, we have the following:

$$\mathbb{E}_0 U^* = \mathbb{E}_0 \int_0^\kappa U^{i,*} di + \mathbb{E}_0 \int_\kappa^1 U^{k,*} dk = (1 - 2\kappa)\mathbb{E}_0 \log R.$$

As this economy is a small open economy,  $R$  is exogenously given. Then clearly, if  $1 - 2\kappa > 0$ ,  $\mathbb{E}_0 U^*$  is concave in  $R$ . Hence, increased volatility of  $R$ -shocks reduces welfare. Similarly, with more general endowment flows, if there are more heavy-saving lenders than light-saving lenders and borrowers together in the economy, then the volatility of  $R$ -shocks will be welfare deteriorating.<sup>7</sup>

### 2.4.2 Binding borrowing limit

Another possible reason why the volatile  $R$ -shocks can be harmful is that there are strict (quantity) limits on borrowing. Suppose the borrower wants to borrow  $d^*$ , but due to the borrowing limit, he can only borrow  $\bar{d} < d^*$ . To simplify the discussion and make the strongest possible case, we assume  $y_1 = 0$  and  $y_2 = 2$  in this whole section. Then the solution (see Eqs. 3 – 6) is:

$$U^{\bar{d}} = \log \bar{d} + \log (2 - R\bar{d}) = \log (2\bar{d} - R\bar{d}^2), \quad (14)$$

for any  $\bar{d} < d^* = \frac{1}{R}$ . From Eq. (14), it is true that with a binding borrowing limit,  $U^{\bar{d}}$  becomes concave in  $R$  as long as  $\bar{d}$  is binding. This shows that as long as borrowing limits

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<sup>6</sup>The ‘distribution effect’ argument clearly does not apply to the case of an individual household in a closed economy.

<sup>7</sup>As we saw in Section 2.3, light-saving lenders will not be hurt by  $R$ -volatility.

remain binding over the range of  $R$ , there is a negative relationship between volatility and welfare. Thus the question of interest is the probability of the cases in which the borrowing limit is binding.

To proceed, we define a critical value of  $R$ ,  $\tilde{R} = \frac{y_2}{2\bar{d}}$ , as the interest factor at which the borrowing constraint is just binding. Remember that  $R$  is assumed to be uniformly distributed over  $[R^*(1 - \sigma), R^*(1 + \sigma)]$ . When  $R$  is above  $\tilde{R}$ , the borrowing constraint does not bind, because with the high  $R$ , the household will not borrow that much anyway. When  $R$  is less than  $\tilde{R}$ , the borrowing constraint binds because with a low  $R$ , the household would like to borrow more. As a result, when the borrowing constraint binds with positive probability, the expected utility is given by

$$\mathbb{E}_0 U^* = \int_{R^*(1-\sigma)}^{\tilde{R}} \frac{\log(2\bar{d} - R\bar{d}^2)}{2R^*\sigma} dR + \int_{\tilde{R}}^{R^*(1+\sigma)} \frac{-\log R}{2R^*\sigma} dR. \quad (15)$$

Note that  $\tilde{R}$  and  $\bar{d}$  are inversely related:  $\tilde{R} = \frac{y_2}{2\bar{d}}$ . When  $\tilde{R}$  is low, it is less likely that the borrowing limit binds because the quantity limit is high. This means that, for this extreme income flow, the relationship between welfare and volatility of  $R$  is still positive. When  $\tilde{R}$  is high, it is more likely that the borrowing limit binds; the relationship between welfare and volatility then becomes negative.

We next define a critical value of  $\tilde{R}$ ,  $\tilde{\tilde{R}}$ , such that, if  $\tilde{R}$  is above this value, the relationship is positive, and if below, the relationship is negative. The critical  $\tilde{\tilde{R}}$  is of particular interest because it determines the lower bound on the probability of the scenarios in which the borrowing limit binds. Again, we use the simulation method to find this critical value. That is, we search over the uniform distribution and check the relationship for each grid point value of  $\tilde{R}$  by using (15). In the numerical exercise, as above, we set the expectation of  $R$  at 1.065, and we obtain the following:

$$\tilde{\tilde{R}} = 1.0651 > R^* = 1.065.$$

So, the probability of a binding borrowing limit is given by

$$P = \int_{R^*(1-\sigma)}^{\tilde{\tilde{R}}} \frac{1}{2\sigma R^*} dR > \int_{R^*(1-\sigma)}^{R^*} \frac{1}{2\sigma R^*} dR = 0.5. \quad (16)$$

Thus, to get a negative relationship, the chance that the borrowing limit binds needs to be high. Equation (16) implies that, in our example, the probability should be greater

than 50%. However, this lower bound of probability is at odds with the data. Mendoza and Smith (2006) gives a probability for a small open economy of hitting a borrowing constraint of 2.45%; and Barro (2008) gives a probability of 2%. As a result, to explain the adverse welfare cost of those external  $R$ -shocks, other mechanisms than the binding borrowing limit need to be incorporated.

## 2.5 Interest rates as a driving force

In the above analysis, there is no real interaction between interest rates and business cycles. The interest rates are not a “true” driving force. One possible remedy for this shortcoming is to extend the discussion to the case where second period income depends on the interest rate. In particular, we assume that

$$y_2 = f(R), \tag{17}$$

$$f'(R) < 0. \tag{18}$$

Equation (17) shows that  $R$  is a driving force of the economy in the sense that the output tomorrow is going to be affected by  $R$ . Equation (18) shows that an increase of  $R$  will lower tomorrow’s output, which is in line with the literature: households will consume relatively more today and invest relatively less for tomorrow, see Uribe and Yue (2006). In this case, the expected utility is given by:

$$\mathbb{E}_0 U^* = \mathbb{E}_0 2 \left[ \log \left( y_1 + \frac{f(R)}{R} \right) - \log 2 \right] + \mathbb{E}_0 \log R. \tag{19}$$

Comparing Eq. (19) to Eq. (7), it is clear that all the qualitative results remain because of Eq. (18). Again, these net borrowers will gain (in ex ante expected value terms) from increased volatility of the interest rate.

## 3 A More Risk Averse CRRA Utility

Another possible reason against the above the positive relationship can be that the utility function is not correctly chosen: households are more risk averse than what the log utility implies. With log utility, the coefficient of risk aversion,  $\gamma$ , is 1, while typically, the estimates of the coefficient of risk aversion are in the range of (0.3,10), see Reinhart

and Vegh (1995). To make households more risk averse, it is natural to consider a general CRRA utility. Accordingly, we modify our preference as:

$$\max_{\{c_1, c_2, d\}} U = \frac{c_1^{1-\gamma} - 1}{1-\gamma} + \frac{c_2^{1-\gamma} - 1}{1-\gamma},$$

subject to (1) and (2). It is straightforward to show the solution is given by

$$c_1^* = \frac{y_1 + y_2/R}{1 + R^{\frac{1}{\gamma}-1}}, \quad (20)$$

$$c_2^* = \frac{y_1 + y_2/R}{\left(1 + R^{\frac{1}{\gamma}-1}\right) / R^{\frac{1}{\gamma}}}, \quad (21)$$

$$d^* = \frac{y_2 - y_1 R^{\frac{1}{\gamma}}}{R + R^{\frac{1}{\gamma}}}, \quad (22)$$

$$U^* = \left[ \frac{y_1 + y_2/R}{\left(1 + R^{\frac{1}{\gamma}-1}\right) / \left(1 + R^{\frac{1}{\gamma}}\right)} \right]^{1-\gamma} / (1-\gamma) - \frac{2}{1-\gamma}. \quad (23)$$

For the purpose of illustration, we again use the simulation, instead of analytical, method to reveal the relationship between the welfare and uncertainty.<sup>8</sup> To simplify the simulation, we assume  $R$  follows a binomial distribution instead of uniform distribution. In particular, we assume there are two possible values of  $R$ :  $R^H$  and  $R^L$ . The probability of each state is 0.5.<sup>9</sup>

### 3.1 Two extreme cases

We first consider the same two extreme scenarios: no endowment today and no endowment tomorrow. To do the simulation, we additionally assume that the coefficient of risk aversion,  $\gamma$ , is set at 2, and the non-stochastic steady state of the interest factor is set at  $R^* = 1.065$ . Figure 2 visually shows the relationship under the two extreme scenarios. The left panel corresponds to the scenario in which  $y_1 = 0$  and  $y_2 = 2$ , while the right panel corresponds to the scenario in which  $y_1 = 2$  and  $y_2 = 0$ . It is clear from

<sup>8</sup>This simulation analysis illustrates the complex relationship between  $y_1$ ,  $y_2$ , and  $\gamma$  in determining the curvature of  $U^*$ . See the Appendix, Section 7.4, for an analytic form of  $\partial^2 U^* / \partial R^2$ .

<sup>9</sup>Originally we assume a uniform distribution. Here we assume a symmetric binomial distribution. The new assumption will not further lose generality because the uniform distribution is also symmetric.

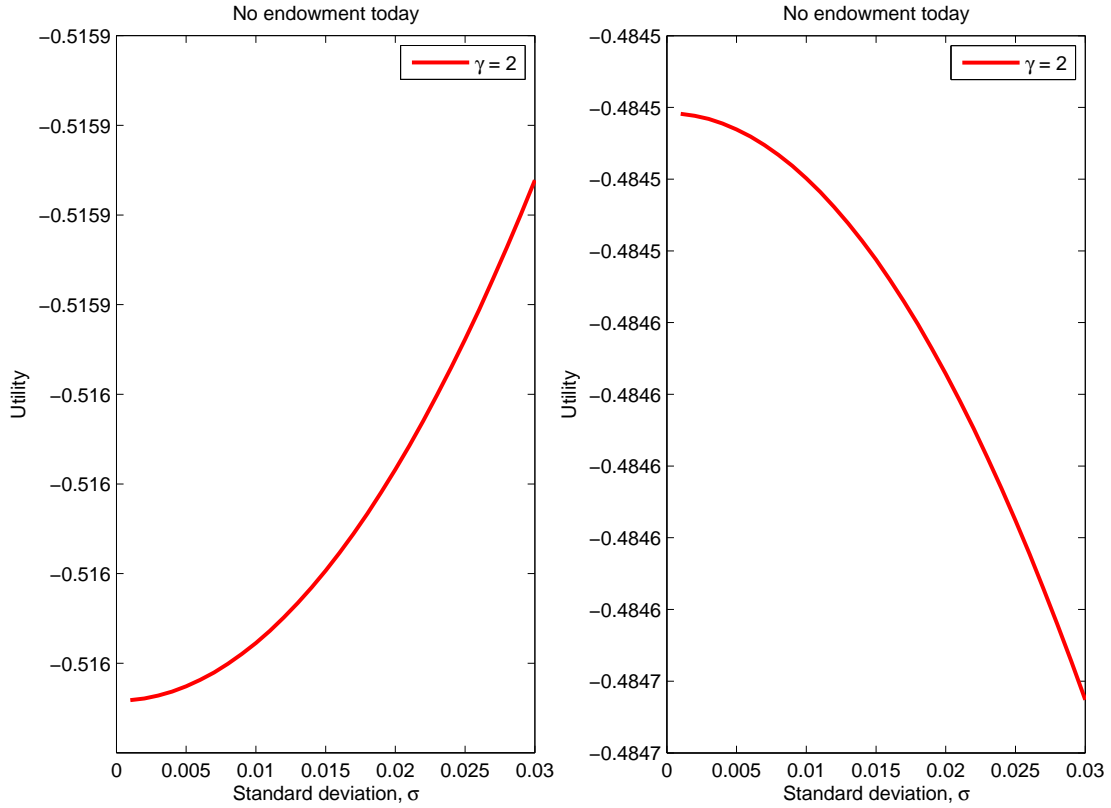


Figure 2: Utility and volatility

the figure that our results in the two extreme cases do not change when we set  $\gamma$  at the usually chosen value 2: the uncertainty of  $R$ -shocks will improve the utility if there is no endowment today; while it will reduce the utility if there is no endowment tomorrow.

In the scenario in which  $y_1 = 0$  and  $y_2 = 2$ , we also have a positive relationship between the volatility of  $R$  and the expected utility when  $\gamma = 2$ . With the same endowment flow, we gradually increase the value of  $\gamma$  above 2 until the relationship turns from positive to negative. By searching over the grid, we find that the critical value associated with this particular endowment flow is  $\gamma^* = 2.73$ . If the coefficient of risk aversion is larger than  $\gamma^*$ , the relationship becomes negative even though there is no endowment today. That is to say, volatility can lower the utility of a borrower if that borrower is sufficiently risk averse and  $\gamma$  is within a reasonable range. Given



that the estimated value of  $\gamma$  ranges from 0.3 to 10 — Uribe and Yue (2006) sets  $\gamma$  at 2; Neumeyer and Perri (2005) sets it at 5; Barro (2008) sets it at 3-4 — this finding appears to provide an answer to our question in Section 2.4. However, as we show next, that low critical value only holds when there is no endowment today. If the endowment today and the endowment tomorrow are sufficiently close, the critical value can be over 80. This degree of risk aversion is far more than any estimates we have seen so far.

### 3.2 Scenarios in between

For all the in-between scenarios, as defined in Section 2.3, we plot the critical values of  $\gamma$  for each pair of  $(y_1, y_2)$  in Fig. 3. For any pair of  $(y_1, y_2)$ , if the true value of  $\gamma$  is above surface, the relationship is negative. If the true value of  $\gamma$  is below, the relationship is positive.

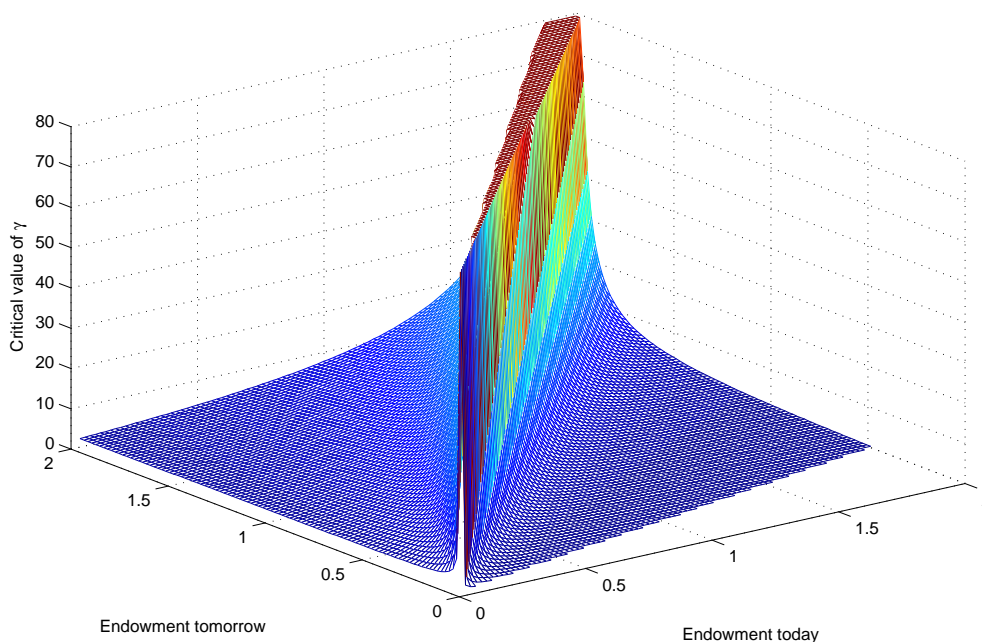


Figure 3: Critical value of  $\gamma$  and endowment flow

From Fig 3 we can see that, if the endowment flow is “smooth”, i.e., the ratio  $x_1/x_2$

is close to 1, it is very likely that volatility is welfare-improving.<sup>10</sup> This is because the critical values of  $\gamma$  required for a negative relationship between  $R$ -volatility and expected utility are very high, well above 10. As a result, if the true value of the coefficient of risk aversion is 5, even though households are quite risk averse, they instead end up benefiting from uncertainty.

This picture provides a possible economic story about why policy makers in a small open economy appear to fail to monitor the economy in good times and why risk aversion households are hurt badly in bad times. Suppose the true value of  $\gamma$  is 5. In good times, endowment flow is smooth, thus the critical value of  $\gamma$  is high, i.e., greater than 5. The policy makers face a very low social welfare cost (or maybe even a welfare benefit) from the uncertainty of  $R$ -shocks. They then tend (boundedly) rationally to ignore those shocks. However, when the economy is hit by a crisis, it is likely that today's endowment unexpectedly falls relatively to tomorrow's endowment. When the endowment flow changes, the critical value of  $\gamma$  can be lowered to well below 5. As a result, the households' utility is significantly lowered, instead of perhaps being enhanced, by the volatility of  $R$ -shocks. Thus households are hurt badly in bad times, and policy makers must then respond to those shocks.

## 4 Habit Formation Utility

Recent literature has begun to use models of habit formation in preferences to study business cycles, asset pricing, and the welfare cost of these cyclical component. [See Uribe and Yue (2006) among many others.] It is thus natural to extend our discussion to the case in which preference incorporates habit formation. Again we use a simple, separable 2-period utility representation,

$$U = \log(c_1 - bc_0) + \log(c_2 - bc_1),$$

where the variable  $b$  denotes the 'habit formation' parameter.<sup>11</sup> As shown in the Appendix (Section 7.5), it is straightforward to solve this consumer's problem with con-

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<sup>10</sup>In the simulation exercise, we capped the maximum value of  $\gamma$  at 80.

<sup>11</sup>This particular functional form has been widely used: in asset pricing literature, see Boldrin et al. (2001), Jermann (1998), and in monetary policy literature, see Christiano et al. (2005).

straints (1) and (2) to obtain the indirect utility function:

$$U^* = \log \left[ \frac{y_2 + Ry_1}{2(b+R)} - \frac{bc_0}{2} \right] + \log \left[ \frac{(y_2 + Ry_1)}{2} + \frac{(bR - b^2)c_0}{2} \right]$$

As above, we first consider the two extreme cases: either no endowment today or no endowment tomorrow. Setting  $c_0 = 0.3$  which is in the range of all possible values for  $c_1$  and  $c_2$ , we have the following picture:

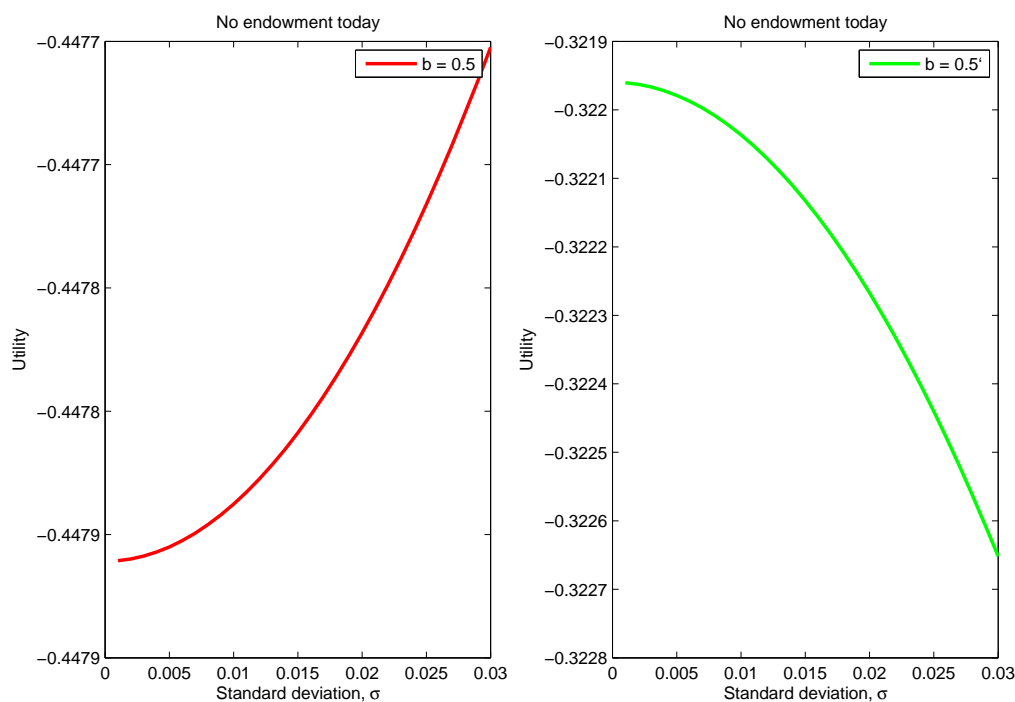


Figure 4: Utility and standard deviation with habit formation

For each given endowment flow,  $(y_1, y_2) \in [0, 2]^2 \setminus \{(0, 0)\}$ , there is a critical value of  $b$  separating welfare enhancing from welfare harming volatility increases. When the true value of  $b$  is above the critical value, increased volatility is welfare improving, and vice versa. In the extreme cases, the critical value associated with the no endowment today (the agent must borrow!) is above 1, while the critical value associated with the no endowment tomorrow (the agent must save!) is below zero.<sup>12</sup> For other endowment

<sup>12</sup>In the former case, consumption must grow substantially for utility to be well defined, while in the latter past consumption compensates for any current consumption drop.

flows, Fig 5 shows the simulation results near the critical value of  $b$ .<sup>13</sup>

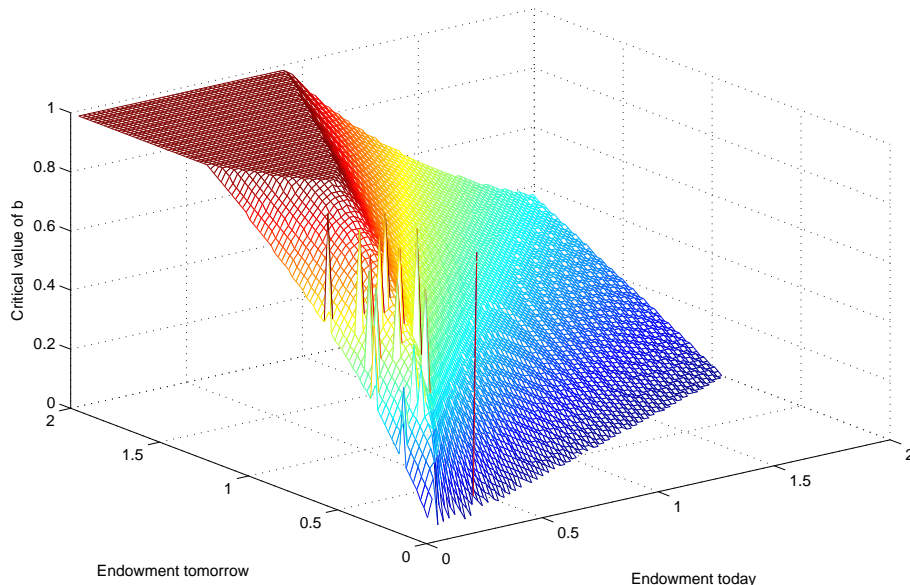


Figure 5: Critical value of  $b$

In the literature, the value of this parameter ranges from 0 to 1. Uribe and Yue (2006) set it at 0.204; Boldrin et al (2001) sets it at 0.73. If the true value of  $b$  is between 0.2 and 0.75, it is then clear from Fig 5 that the welfare cost of volatility is highly sensitive to the endowment flow. If tomorrow’s endowment is very far below today’s endowment, it is very likely that households dislike volatility, while if tomorrow’s endowment is far more than today’s, it is almost certain that households like volatility.

This shows that “habit formation” in the households’ preferences further complicates the welfare analysis in a small open economy. To make the point as straightforwardly as possible, we focus on the those “smooth” endowment flows in which  $y_1$  is very close to  $y_2$ . In the discussion about the general CRRA preferences, we showed that if the endowments are “smooth,” it is very likely that households will be better off if the  $R$ -volatility is higher because the threshold values for  $\gamma$  are high, as shown in Fig. 3. However, once the habit parameter is introduced, households who put a lot of weight on the past consumption, i.e., for whom the value of  $b$  is high, will be worse off with the

<sup>13</sup>In the simulation exercise, we capped the maximum value of  $b$  at 1.

higher  $R$ -volatility even if the endowment flows are “smooth,” and households are mildly risk averse, e.g., the coefficient of risk aversion is set at 1 above. Clearly, serious policy analysis needs to take the possibility of habit persistence into consideration, particularly if such persistency is important in fitting the data on which the analysis is built.

## 5 Conclusion

In this paper we analyze the welfare impact of the volatility of external  $R$ -shocks on a representative household in a small open economy. Contrary to the conventional belief, the volatility of  $R$ -shocks itself does not reduce ex-ante welfare. When there is a borrowing limit that binds with a sufficiently high probability, this volatility becomes, however, detrimental. Further, a positive relationship holds between volatility and ex-ante welfare for three widely used preference representations, and with a wide range of values of the relevant parameters.

These results can have important policy implications for small open economies. For example, our results show that it is not universally true that risk averse households will be worse off with more volatile external, exogenous interest rate shocks. Even though there is no direct discussion about policy, we suspect that our results imply that counter-cyclical policy is not always preferred to the procyclical policy when the shocks faced by a small open economy are predominantly interest-rate shocks, a consequence of the positive relationship between welfare and volatility we have shown.

More importantly, our results suggest the optimality of policy crucially depends on the structure of the endowment flow, as well as the true level of risk aversion, and/or the strength of habit formation. As a result, for serious policy analysis, the effect of any policy on the endowment flow must be taken into consideration, and the accuracy and the confidence interval of the estimates of those structural parameters need to be carefully examined.

However, to incorporate the two mentioned points requires a substantial extension of the model, which will be undertaken in future research. As is well known, interest rate volatility also affects production more profoundly than just changing the income flows. Since the current model is of course only part of the story of the welfare impact of interest rate volatility, we will continue this research by extending the analysis from an endowment economy to a production economy.

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## 7 Appendix

### 7.1 The analytical expression of Eq. (13)

From Eq. (13), we have:

$$\begin{aligned}
\mathbb{E}_0 U^* &= \mathbb{E}_0 2 \log \left( x_1 + \frac{x_2}{R} \right) + \mathbb{E}_0 \log R, \\
&= \frac{1}{2R^*\sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} [2 \log (x_1 R + x_2) - \log R] dR, \\
&= \frac{1}{R^*\sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log (x_1 R + x_2) dR - \frac{1}{2R^*\sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log R dR,
\end{aligned}$$

Let's define  $Q = x_1 R + x_2$ , then the first term can be written as

$$\begin{aligned}
&\frac{1}{R^*\sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log (x_1 R + x_2) dR \\
&= \frac{1}{R^*\sigma x_1} \int_{R^*(1-\sigma)x_1+x_2}^{R^*(1+\sigma)x_1+x_2} \log Q dQ = \frac{1}{R^*\sigma x_1} \int_{\underline{Q}}^{\bar{Q}} \log Q dQ \\
&= \frac{Q \log Q}{R^*\sigma x_1} \Big|_{\underline{Q}}^{\bar{Q}} - \frac{1}{R^*\sigma x_1} \int_{\underline{Q}}^{\bar{Q}} dQ = \frac{\bar{Q} \log \bar{Q} - \underline{Q} \log \underline{Q}}{R^*\sigma x_1} - \frac{\bar{Q} - \underline{Q}}{R^*\sigma x_1} \\
&= \frac{\bar{Q} \log \bar{Q} - \underline{Q} \log \bar{Q} + \underline{Q} \log \bar{Q} - \underline{Q} \log \underline{Q}}{R^*\sigma x_1} - 2 \\
&= 2 \log \bar{Q} + \frac{\underline{Q} \log \bar{Q} - \underline{Q} \log \underline{Q}}{R^*\sigma x_1} - 2 \\
&= 2 \log \bar{Q} + \frac{\underline{Q}}{R^*\sigma x_1} \log \left[ \frac{R^*(1+\sigma)x_1+x_2}{R^*(1-\sigma)x_1+x_2} \right] - 2
\end{aligned}$$

where

$$\begin{aligned}
\bar{Q} &= R^*(1+\sigma)x_1+x_2 \\
\underline{Q} &= R^*(1-\sigma)x_1+x_2
\end{aligned}$$

The second term can be expressed as

$$\begin{aligned}
\frac{1}{2R^*\sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log R dR &= \frac{1}{2R^*\sigma} \left[ R \log R \Big|_{R^*(1-\sigma)}^{R^*(1+\sigma)} \right] - 1 \\
&= \log \bar{R} + \frac{\underline{R}}{2R^*\sigma} \log \left[ \frac{R^*(1+\sigma)}{R^*(1-\sigma)} \right] - 1
\end{aligned}$$



where

$$\begin{aligned}\bar{R} &= R^*(1 + \sigma) \\ \underline{R} &= R^*(1 - \sigma)\end{aligned}$$

So, the expected utility can be rewritten as

$$\begin{aligned}\mathbb{E}_0 U^* &= \frac{1}{R^* \sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log(x_1 R + x_2) dR - \frac{1}{2R^* \sigma} \int_{R^*(1-\sigma)}^{R^*(1+\sigma)} \log R dR, \\ &= 2 \log \bar{Q} + \frac{Q}{R^* \sigma x_1} \log \left[ \frac{R^*(1 + \sigma)x_1 + x_2}{R^*(1 - \sigma)x_1 + x_2} \right] \\ &\quad - \left\{ \log \bar{R} + \frac{\underline{R}}{2R^* \sigma} \log \left[ \frac{R^*(1 + \sigma)}{R^*(1 - \sigma)} \right] \right\} - 1 \\ &= 2 \log [R^*(1 + \sigma)x_1 + x_2] + \frac{R^*(1 - \sigma)x_1 + x_2}{R^* \sigma x_1} \log \left[ \frac{R^*(1 + \sigma)x_1 + x_2}{R^*(1 - \sigma)x_1 + x_2} \right] \\ &\quad - \left\{ \log [R^*(1 + \sigma)] + \frac{(1 - \sigma)}{2\sigma} \log \left[ \frac{R^*(1 + \sigma)}{R^*(1 - \sigma)} \right] \right\} - 1\end{aligned}$$

## 7.2 The analytical expression of Eq. (15)

We start with Eq. (15)

$$\mathbb{E}_0 U^* = \int_{R^*(1-\sigma)}^{\bar{R}} \frac{\log(2\bar{d} - R\bar{d}^2)}{2R^* \sigma} dR + \int_{\bar{R}}^{R^*(1+\sigma)} \frac{-\log R}{2R^* \sigma} dR. \quad (24)$$

Define  $M = y_2 \bar{d} - R\bar{d}^2$ . Thus, the first term can be rewritten as:

$$\begin{aligned}& \int_{R^*(1-\sigma)}^{\bar{R}} \frac{\log(2\bar{d} - R\bar{d}^2)}{2R^* \sigma} dR \\ &= \frac{-1}{2R^* \sigma} \int_{\underline{M}}^{\tilde{M}} \left[ \frac{\log M}{\bar{d}^2} \right] dM \\ &= \frac{-1}{2R^* \sigma \bar{d}^2} \left[ \tilde{M} \log \tilde{M} - \underline{M} \log \underline{M} - (\tilde{M} - \underline{M}) \right]\end{aligned}$$

where

$$\underline{M} = 2\bar{d} - \underline{R}\bar{d}^2 \quad (25)$$

$$\tilde{M} = 2\bar{d} - \bar{R}\bar{d}^2 \quad (26)$$

The second term is:

$$\int_{\tilde{R}}^{R^*(1+\sigma)} \frac{-\log R}{2R^*\sigma} dR = \frac{-1}{2R^*\sigma} \left[ \bar{R} \log \bar{R} - \tilde{R} \log \tilde{R} - (\bar{R} - \tilde{R}) \right].$$

### 7.3 Solution when utility function is a general CRRA

The Euler equation is given by

$$\frac{c_1^{-\gamma}}{c_2^{-\gamma}} = R \Rightarrow \frac{c_2}{c_1} = R^{\frac{1}{\gamma}}$$

Thus we have

$$\frac{y_2 - Rd}{y_1 + d} = R^{\frac{1}{\gamma}} \Rightarrow d^* = \frac{y_2 - y_1 R^{\frac{1}{\gamma}}}{R + R^{\frac{1}{\gamma}}} \quad (27)$$

$$c_1^* = y_1 + d^* = \frac{y_1 + y_2/R}{1 + R^{\frac{1}{\gamma}-1}} \quad (28)$$

$$c_2^* = c_1^* R^{\frac{1}{\gamma}} = \frac{y_1 + y_2/R}{(1 + R^{\frac{1}{\gamma}-1}) / R^{\frac{1}{\gamma}}} \quad (29)$$

$$\begin{aligned} V^* &= \frac{(c_1^*)^{1-\gamma} - 1}{1-\gamma} - \frac{(c_2^*)^{1-\gamma} - 1}{1-\gamma} = \frac{(c_1^*)^{1-\gamma}}{1-\gamma} \left(1 + R^{\frac{1}{\gamma}}\right)^{1-\gamma} - \frac{2}{1-\gamma} \\ &= \frac{\left(\frac{y_1 + y_2/R}{1 + R^{\frac{1}{\gamma}-1}}\right)^{1-\gamma}}{1-\gamma} \left(1 + R^{\frac{1}{\gamma}}\right)^{1-\gamma} - \frac{2}{1-\gamma} \\ &= \left[ \frac{y_1 + y_2/R}{(1 + R^{\frac{1}{\gamma}-1}) / (1 + R^{\frac{1}{\gamma}})} \right]^{1-\gamma} / (1-\gamma) - \frac{2}{1-\gamma} \end{aligned} \quad (30)$$

## 7.4 Curvature of CRRA expected utility

Letting  $w(R) = y_1 + y_2/R$ , we have:

$$\frac{\partial^2 U^*}{\partial R^2} = -\frac{1}{R^5 \left(1 + R^{\frac{1-\gamma}{\gamma}}\right)^3} \left( \frac{\left(1 + R^{\frac{1}{\gamma}}\right) R w(R)}{R + R^{\frac{1}{\gamma}}} \right)^{-\gamma} \times \left\{ \begin{aligned} & \frac{2(\gamma-1)R^{\frac{1}{\gamma}} \left(1 + R^{\frac{1}{\gamma}}\right) \left(R + R^{\frac{1}{\gamma}}\right) y_2}{\gamma} + \frac{2R^{\frac{1}{\gamma}} \left(R + R^{\frac{1}{\gamma}}\right)^2 y_2}{\gamma} - 2 \left(1 + R^{\frac{1}{\gamma}}\right) \left(R + R^{\frac{1}{\gamma}}\right)^2 y_2 - \\ & \frac{2(\gamma-1)^2 R^{\frac{2}{\gamma}} \left(1 + R^{\frac{1}{\gamma}}\right) R w(R)}{\gamma^2} - \frac{2(\gamma-1) R^{\frac{2}{\gamma}} \left(R + R^{\frac{1}{\gamma}}\right) R w(R)}{\gamma^2} + \\ & \frac{(\gamma-1)(2\gamma-1) R^{\frac{2}{\gamma}} \left(1 + R^{\frac{1}{\gamma}}\right) \left(R + R^{\frac{1}{\gamma}}\right) R w(R)}{\gamma^2} - \frac{(\gamma-1) R^{\frac{1}{\gamma}} \left(R + R^{\frac{1}{\gamma}}\right)^2 R w(R)}{\gamma^2} + \\ & \frac{\left( \left( R^{2+\frac{1}{\gamma}} + \gamma R^{\frac{2+\gamma}{\gamma}} \right) y_1 + (\gamma-1) R^{1+\frac{1}{\gamma}} (y_1 - y_2) - \left( \gamma R + R^{\frac{1}{\gamma}} \right) y_2 \right)}{\gamma \left(1 + R^{\frac{1}{\gamma}}\right) R w(R)} \end{aligned} \right\}.$$

We explore the convexity properties of this function through simulation.

## 7.5 Solution when utility function is habit formation

The Lagrangian is given by

$$L = \log(c_1 - bc_0) + \log(c_2 - bc_1) + \lambda_1(y_1 + d - c_1) + \lambda_2(y_2 - Rd - c_2)$$

The optimality conditions are given by

$$\begin{aligned} \frac{1}{c_1 - bc_0} - \frac{b}{c_2 - bc_1} &= \lambda_1, \\ \frac{1}{c_2 - bc_1} &= \lambda_2, \\ \lambda_1 &= \lambda_2 R. \end{aligned}$$

Combine the three optimality conditions, we obtain the following

$$\begin{aligned} \frac{1}{c_1 - bc_0} - \frac{b}{c_2 - bc_1} &= \frac{R}{c_2 - bc_1} \\ \frac{1}{c_1 - bc_0} &= \frac{b + R}{c_2 - bc_1} \end{aligned}$$

Thus we have

$$\begin{aligned}(b + R)(c_1 - bc_0) &= (c_2 - bc_1) \\ (2b + R)(c_1) &= c_2 + b(b + R)c_0\end{aligned}$$

Plug in the period resource constraint, we get

$$\begin{aligned}(2b + R)(y_1 + d) &= y_2 - Rd + b(b + R)c_0 \\ 2(b + R)d &= y_2 - (2b + R)y_1 + b(b + R)c_0 \\ d^* &= \frac{y_2 - (2b + R)y_1}{2(b + R)} + \frac{bc_0}{2}\end{aligned}$$

The solution to consumption is then given by

$$\begin{aligned}c_1^* &= y_1 + d^* = y_1 + \frac{y_2 - (2b + R)y_1 + b(b + R)c_0}{2(b + R)} = \frac{y_2 + Ry_1}{2(b + R)} + \frac{bc_0}{2} \\ c_2^* &= y_2 - Rd^* = y_2 - \frac{y_2 - (2b + R)y_1 + b(b + R)c_0}{2(b + R)}R = \frac{(2b + R)(y_2 + Ry_1)}{2(b + R)} + \frac{bRc_0}{2} \\ U^* &= \log(c_1^* - bc_0) + \log(c_2^* - bc_1^*) \\ &= \log\left[\frac{y_2 + Ry_1}{2(b + R)} + \frac{bc_0}{2} - bc_0\right] + \log\left[\frac{(2b + R)(y_2 + Ry_1)}{2(b + R)} + \frac{bRc_0}{2} - b\left(\frac{y_2 + Ry_1}{2(b + R)} + \frac{bc_0}{2}\right)\right] \\ &= \log\left[\frac{y_2 + Ry_1}{2(b + R)} - \frac{bc_0}{2}\right] + \log\left[\frac{(y_2 + Ry_1)}{2} + \frac{(bR - b^2)c_0}{2}\right]\end{aligned}$$