Visualizing and Testing Convergence Between Two Income Distributions

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Abstract

We use the interdistributional Lorenz curves (ILCs) of Butler and McDonald (1987) to visualize convergence or divergence between income distributions. To illustrate the idea, we compare income distributions from Spain, Italy, and Germany. We also offer methods to test for significant differences between the 45-degree line and an ILC, or between ILCs in different years. The tests apply to any partial moment of the distributions and impose no prior restrictions on the functional form of the underlying distribution. We illustrate the statistical inference tests by an application to income distributions for whites and nonwhites in the United States.

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Introduction

In a variety of applications, the evaluation of economic policies depends on whether two income distributions are becoming more alike, or converging. Have the policies of the European Union led to convergence among the income distributions of member countries? Have the policies of the United States made distributions of income by race more alike over time? In the making and reporting of such evaluations, it would be useful to have a visual representation of convergence and a means for testing whether any movement is statistically significant. We illustrate how to use the interdistributional Lorenz curves (ILCs) of Butler and McDonald (1987) to represent convergence visually, and show how to test for significant movements in these curves over time.

The familiar Lorenz curve offers a visual representation of the dispersion of incomes *within one* distribution. An ILC looks at inequality *across two* distributions. Corrado Gini (1916, 1959), who proposed the most famous index of inequality derived from the Lorenz curve, also proposed different ways to capture the degree of inequality across two distributions. The first way measures the extent of overlapping between two distributions and the second way measures the difference between the concentrations of each distribution below a reference point in the other. The ILC corresponds in one of its forms to the second approach, as Deutsch and Silber (1997) have demonstrated. Deutsch and Silber (1999) survey the literature on inequality across distributions, giving attention

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to the development of the concepts of the economic distance between two subgroups and of the economic advantage by one subgroup over another. They also associate the ILCs of Butler and McDonald (1987) with the latter idea.

Dagum (1980) defined economic distance ratios that measure the degree of economic affluence of a richer subgroup relative to a poorer one, but Shorrocks (1982) criticized their formulation. Dagum (1987) proposed a reformulation, but Vinod (1985) pursued a new direction by defining measures of economic advantage by one subgroup over another. Gastwirth (1985) questioned whether these measures were substantively different from others already available. Butler and McDonald (1987) took a stochastic dominance approach to measuring economic advantage, which yields the ILCs that we use in this paper.[2]

The earlier approaches to interdistributional inequality collapsed all the information in the distributions into a summary measure (index number), as the Gini coefficient reduces the information in a Lorenz curve to a single number. While such an approach is convenient, it also imposes value judgments that may be neither obvious nor widely accepted when made explicit. Even Butler and McDonald (1987) collapse their ILCs into Peitra indices in their application. This paper shows how to implement the dominance approach in two applications.

With suitable data, it is possible to use statistical inference procedures in comparisons of ILCs. Our approach to statistical inference covers any incomplete

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[2] A related literature in sociology, starting with Yitzhaki and Lerman (1991) and continuing through Reardon, *et. al.* (2006), develops measures of income (and other forms of) stratification or segregation. Yitzhaki and Lerman (1991) investigate the relationship between inequality and stratification. We compare our contribution to Reardon, *et. al.* (2006) later in the paper. Like the literature on economic distance and economic advantage, studies of stratification or segregation rely primarily on index numbers rather than dominance comparisons.

moment, and it imposes no prior restrictions on the functional form of the underlying distribution. We can test whether an ILC for a given year differs significantly from the 45-degree line, and whether ILCs for two years differ significantly. The latter enable us to determine whether ILCs are converging toward, or diverging from, the 45-degree line. Bishop, *et. al.* (2003) generate ILCs for one moment (income shares), but do not provide statistical inference procedures. Bishop, *et. al.* (2004) offer procedures for comparisons of income shares *within* a given year, but no rigorous test for comparisons *across* years. This paper provides a more comprehensive presentation by including comparisons for *any moment* (population shares, income shares, etc.) as well as comparisons across groups *and* over time.

The next section describes the construction of ILCs with an application to European convergence. We then present the statistical inference procedures with an application to economic advantage by race in the United States. The final section offers a summary of our main findings, along with concluding comments.

The Construction of Interdistributional Lorenz Curves

As noted above, interdistributional inequality has roots in a proposal by Gini (1916) to measure the inequality across two distributions by the difference between the concentrations of each distribution below a reference point in the other. In the spirit of a stochastic dominance approach, we alter this proposal to include more than one reference point and we select common reference points from the pooled distribution of incomes for the two groups. We obtain the concentrations of the two distributions below the common reference points from partial moments of the distributions, as we show formally below.

Intuitively, the first partial moment yields the proportion of *people* with incomes below a given reference point, and the second partial moment yields the proportion of *total income* that falls below a given reference point.

To perform a comparison of inequality across two distributions, we must first partition the population by some socio-economic characteristic (such as region, race, or gender), and compare the "degree of affluence" (Dagum, 1980) of one subgroup relative to another. ILCs plot partial moments for each subgroup at the common reference points, or "income targets". We can construct ILCs for each partial moment, but we focus on the first and second moments, that is, the population and income shares for each group below the set of common targets, respectively. If the moments are equal at each target, the ILC lies on the 45-degree line. If the moments are not equal, and we assign the group with larger income shares to the horizontal axis, the ILC lies below the 45-degree line, like a Lorenz curve.

We begin by formally defining the concepts underlying ILCs. Let x be a continuous income variable with a probability density $f(x)$. Let $F(x)$ denote the cumulative distribution function (CDF) of x, and let the inverse CDF of x be written $0 \le F^{-1}(p) \le \infty$. Without loss of generality, let $\tau = F^{-1}(p)$ define the target incomes. When p = 0.1, 0.2, …, 1.0, the target incomes become the decile order statistics. Let I_τ^x be an indicator variable such that $I_\tau^x = 1$ if $x \le \tau$ and $I_\tau^x = 0$ otherwise.

Given a target income τ, we can define the *h*-th partial moment for $x < \tau$ of the density function $f(x)$ as

(1) $$M(\tau; h, x) = \int_0^\tau x^h f(x)dx = \int_0^\infty (xI_\tau^x)^h dF(x) = E\left[(xI_\tau^x)^h\right],$$

where E is the expectation operator. For $h = 0$ the partial moment reduces to $F(\tau)$, which gives the cumulative population share with incomes below the income target τ. Following Butler and McDonald (1987), we define the normalized incomplete moment of x for $x \le \tau$ as

(2) $$\phi(\tau, h, x) = M(\tau, h, x) / E(x^h),$$

For $h = 1$, the normalized incomplete moment gives Lorenz ordinates, $\phi(\tau,1,x) = L(\tau; x)$, which becomes clear if we write the Lorenz curve in the form proposed by Bishop, Chow, and Formby (1994),

(3) $$L(\tau; x) = \mu_x^{-1} \int_0^{\tau} x f(x) dx = \mu_x^{-1} \int_0^{\infty} x I_\tau^x dF(x) = E\left[x I_\tau^x\right] / E(x),$$

where $E(x)$ is the mean of x. We can interpret $L(\tau; x)$ as the proportion of total income received by individuals with incomes $x \le \tau$, the target income.

To represent population subgroups, let incomes be classified by K mutually exclusive groups $\{\Phi_k, k = 1,2,..,K\}$ and define an indicator variable G_k^x such that $G_k^x = 1$ if $x \in \Phi_k$ and $G_k^x = 0$ otherwise. This indicator variable allows us to rewrite (3), because $E(x I_\tau^x | G_k^x = 1) = E(x G_k^x I_\tau^x) / E(G_k^x)$ and $E(x | G_k^x = 1) = E(x G_k^x) / E(G_k^x)$. Bishop, Chow, and Zeager (2002) use this approach to show that:

THEOREM 1. $L(\tau, x)$ *can be decomposed by* $L(\tau, x^{(k)})$ *for* $k = 1, 2, \ldots, K$ *in that*

(4) $$L(\tau, x) = \sum_{k=1}^{K} P^{(k)} \cdot L(\tau, x^{(k)}),$$

where $P^{(k)} = E\left[x \cdot G_k^x\right] \big/ E(x)$. We can interpret $P^{(k)}$ as the income share of subgroup k with respect to the income variable x.

We show here that similar reasoning can be applied to cases in which $h \neq 1$. That is, expression (2) can also be decomposed by population group k (k = 1, 2, …, K):

$$\text{(5)} \qquad \phi(\tau, h, x^{(k)}) = M(\tau, h, x^{(k)}) / E\left[(x \cdot G_k^x)^h\right] = E\left[(x \cdot G_k^x \cdot I_\tau^x)^h\right] / E\left[(x \cdot G_k^x)^h\right].$$

Since $\phi(\tau, h, x^{(k)})$ is expressed as the expected value of a function of the random income times the indicator variables for targets and population groups, we can easily determine the property of its decomposition from the overall $\phi(\tau, h, x)$ as follows:

THEOREM 2. *$\phi(\tau, h, x)$ can be decomposed by $\phi(\tau, h, x^{(k)})$ for k = 1, 2, …, K in that*

$$\text{(6)} \qquad \phi(\tau, h, x) = \sum_{k=1}^{K} w^{(k)} \cdot \phi(\tau, h, x^{(k)}),$$

where $w^{(k)} = E\left[(x \cdot G_k^x)^h\right] \big/ E(x^h)$. We can interpret $w^{(k)}$ as the h-th moment for subgroup k with respect with the income variable x.

Butler and McDonald (1987) use $\phi(\tau, h, x)$ to define two "natural" ILCs for population subgroups. For ease of presentation, consider two population subgroups, so that K = 2. The first natural ILC is obtained by plotting $\phi(\tau, 0, x^{(1)})$ against $\phi(\tau, 0, x^{(2)})$. With h = 0, we are plotting population (instead of income) shares below common income targets in both groups. If the population shares are equal across the groups, $\phi(\tau, 0, x^{(1)})$ = $\phi(\tau, 0, x^{(2)})$ at each target income (τ), and the ILC corresponds to the 45-degree line. On the other hand, if one group (say, k = 1) is "disadvantaged" [$\phi(\tau, 0, x^{(1)}) > \phi(\tau, 0, x^{(2)})$ at each τ], and that group is assigned to the horizontal axis, then the ILC lies below the 45-degree line, like the Lorenz curve. In those cases for which a group is disadvantaged at some income targets and advantaged at others, the ILC crosses the 45-degree line. The second natural ILC is obtained by setting $h = 1$ (yielding income, instead of population shares), plotting $\phi(\tau,1,x^{(1)})$ against $\phi(\tau,1,x^{(2)})$, and interpreting it in similar fashion.

Normatively, ILC dominance implies that a subgroup of the population with either a larger share of its people or a larger share of its incomes below a give income level is "disadvantaged" relative to the rest of the population. Defining the reference points as income levels, instead of percentages of the population (as with Lorenz curves, which depend totally on relative income comparisons), makes it possible to appeal to the "more is better" principle that has very broad support among economists. Further, we note that a mean-preserving transformation within any sub-group does not reduce its "economic disadvantage."

Figure 1 illustrates a hypothetical ILC for two population subgroups, A and B. Points on the ILC are generated as follows. We select the order statistics (upper income cutoffs) for the deciles in the distribution of pooled incomes (across subgroups) as target incomes and estimate each ILC at these reference points. For each target income, we plot the corresponding shares for the subgroups A and B on the vertical and horizontal axes, respectively. Therefore, the B-A difference (economic advantage) will be the vertical distance between the 45-degree line and the ILC.

We illustrate actual ILCs (for the case $h = 1$) with Luxembourg Income Study (LIS) data on European countries. The purpose of these illustrations is to show the three possible outcomes that one can obtain from comparisons of ILCs: dominance, equality, and crossing. The LIS provides micro data on per capita, disposable family income for Germany in 1989 and 2000, for Spain in 1990 and 2000, and for Italy in 1991 and 2000, given in terms of each country's currency.[3] To make the income *levels* comparable across countries and consistent over time, we rescale the LIS incomes using the real, per capita GDP of the countries from the Penn World Tables in 1990 (Spain = $12,525, Italy = $16,817, and Germany = $16,947) and 2000 (Spain = $19,037, Italy = $22,867, and Germany = $23,917). Even with the rescaling, the LIS micro data determine the *dispersion* of incomes within each country.

Figure 2 plots the income shares for Spain against those for Italy, with Spain measured on the horizontal axis and Italy on the vertical axis. ILCs for the earlier and later years appear on the same diagram. Given that the ILCs for each year lie below the 45-degree line, the income shares below each income target are smaller for Italy than for Spain, so Spain is clearly the disadvantaged group in both years. The 2000 ILC also lies everywhere above the ILC for the earlier years, so it appears that a convergence toward the 45-degree line occurred during the 1990s. That is, the income distributions for the two countries apparently became more alike.

Figure 3 shows the corresponding comparison for Germany and Spain, with Germany measured on the vertical axis. As in the previous comparison, the initial ILC lies everywhere below the 45-degree line, indicating an economic advantage for Germany over Spain. Yet the 2000 ILC lies at or above the ILC for the earlier years, which implies that Germany's advantage diminished during the 1990s. Here the picture is more subtle, however, than in the previous comparison. At the bottom of the distributions we see no

[3] The LIS database supports a limited number of software applications and, unlike most databases, is not "downloadable." Given these limitations, we defer our presentation of statistical inference procedures to the following section.

change, whereas at the top of the distributions the economic advantage seems to have vanished entirely.

Figure 4 shows the remaining comparison of Italy and Germany, with Italy measured on the horizontal axis and Germany on the vertical axis. In contrast to the previous comparisons, the ILCs both cross the 45-degree line, so we do not find a clear economic advantage. Germany may have a slight advantage at the bottom of the income distributions and Italy may have a slight advantage at the top. Further evaluation of these "slight advantages" observed in Figure 4 calls for more appropriate statistical inference procedures, which we present in the next section.

Statistical Inference Procedures

To develop an inference test for ILCs, we select a set of m income classes or target income levels, denoted by $\{\tau_i | i = 1,2,...,m\}$, to which there correspond K sets of ILC ordinates $\{\phi(\tau_i, h, x^{(k)}) \mid i = 1, 2, \ldots, m, \text{ and } k = 1,2,\ldots,K\}$. This approach allows us to relax the assumption of a continuous CDF, as the Lorenz and concentration ordinates correspond to a set of target incomes instead of a set of quantile functions. Empirically, the targets are selected as a set of sample quantiles ($\hat{\xi}_p$) of the income variable x, i.e., $p_1 = 0.1$, $p_2 = 0.2$, …, $p_9 = 0.9$, which in our application are sample deciles. Then, if we draw a random sample of size N from the population, and if the CDF of x is strictly monotonic, $\hat{\xi}_p$ has the property of strong or almost sure consistency (Rao 1965, 335).

Let $(x_1, x_2, \ldots, x_N)$ be a set of identical and independently distributed (i.i.d.) random sample incomes drawn from the population density *f(x)*. According to equation (5), the decomposed interdistributional Lorenz ordinates can be estimated as

$$(7) \qquad \hat{\phi}^h_{i,(k)} = \hat{\phi}(\tau_i, h, x^{(k)}) = \left[N^{-1} \sum_{j=1}^{N} (x_j G_k^{x_j} I_{\tau_i}^{x_j})^h \right] \Big/ \left[N^{-1} \sum_{j=1}^{N} (x_j G_k^{x_j})^h \right]$$

Let $\underset{1x(mK)}{\Phi} = (\Phi_1, \Phi_2, ..., \Phi_{mK})' = \left((\phi^h_{1,(1)}, \ldots, \phi^h_{m,(1)}), (\phi^h_{1,(2)}, \ldots, \phi^h_{m,(2)}), \ldots, (\phi^h_{1,(K)}, \ldots, \phi^h_{m,(K)}) \right)'$

be a vector of *mK* decomposed ILC ordinates. The estimates of the vector Φ can be written as $\underset{1x(mK)}{\hat{\Phi}} = (\hat{\Phi}_1, \hat{\Phi}_2, ..., \hat{\Phi}_{mK})' = \left((\hat{\phi}^h_{1,(1)}, \ldots, \hat{\phi}^h_{m,(1)}), (\hat{\phi}^h_{1,(2)}, \ldots, \hat{\phi}^h_{m,(2)}), \ldots, (\hat{\phi}^h_{1,(K)}, \ldots, \hat{\phi}^h_{m,(K)}) \right)'$. From equations (5) through (7), the decomposed ILC ordinates are functions of $E\left[(xG_k^x I_{\tau_i}^x)^h\right]$ and $E\left[(xG_k^x)^h\right]$ for $i = 1,2, \ldots, m$ and $k = 1,2,\ldots, K$. To derive the asymptotic sampling distribution of $\overline{\Phi}$, it is necessary to determine the sampling distributions of these estimates, $\overline{(xG_k^x I_{\tau_i}^x)^h}$,and $\overline{(xG_k^x)^h}$ for $i = 1,2, \ldots, m$ and $k = 1,2,\ldots, K$.

We define the vector of $K(m+1)$ parameter estimators as

$$\underset{1x[K(m+1)]}{\overline{\Psi}} = (\overline{\psi}_1, \overline{\psi}_2, ..., \overline{\psi}_{K(m+1)})' = \left(\left(\overline{(xG_1^x I_{\tau_1}^x)^h}, \ldots, \overline{(xG_1^x I_{\tau_m}^x)^h}, \overline{(xG_1^x)^h} \right), \ldots, \left(\overline{(xG_K^x I_{\tau_1}^x)^h}, \ldots, \overline{(xG_K^x I_{\tau_m}^x)^h}, \overline{(xG_K^x)^h} \right) \right)'.$$

THEOREM 3. *Suppose* $(x_1, x_2, ..., x_N)$ *are i.i.d. random samples of a size of N drawn from the population density function f(x). Given a set of predefined target incomes* $\{\tau_i \,|\, i = 1,2,3,..., m\}$ *such that* $0 < \tau_1 < \ldots < \tau_m < \infty$, *and a population decomposed into K mutually exclusive groups, the vector* $\sqrt{N}(\overline{\Psi} - \Psi)$ *converges in probability to a K(m+1) variate normal distribution with mean zero and a variance-covariance* $\Omega = (\sigma_{i,j})$, *where*

$$\underset{(K(m+1))x(K(m+1))}{\Omega} = \begin{bmatrix} \left[\alpha_{ij}^{11}\right] & \cdots & \left[\alpha_{ij}^{1K}\right] \\ \vdots & \ddots & \vdots \\ \left[\alpha_{ij}^{K1}\right] & \cdots & \left[\alpha_{ij}^{KK}\right] \end{bmatrix}, \textit{ and}$$

$$\left[\alpha_{ij}^{kl}\right]_{(m+1)x(m+1)} = \begin{cases} Cov[(xG_k^x I_{\tau_i}^x)^h, (xG_l^x I_{\tau_j}^x)^h] \text{ for } i, j \le m \\ Cov[(xG_k^x I_{\tau_i}^x)^h, (xG_l^x)^h] \text{ for } i \le m, \quad j = (m+1) \\ Cov[(xG_k^x)^h, (xG_l^x I_{\tau_j}^x)^h] \text{ for } i = (m+1), \ j \le m \\ Cov[(xG_k^x)^h, (xG_l^x)^h] \text{ for } i = j = (m+1) \end{cases},$$

where *Cov* denotes the covariance measure.

PROOF. Given that the income samples *x* and the indicator variables *G* and *I* are *i.i.d.*, the *h*-th power function *i.i.d.* random variable is also *i.i.d.* From direct calculations, it can be shown that $E(\overline{\psi}_i) = \psi_i$, $i = 1,2,..., K(m+1)$. Then, for large samples, the Kolmogorov Strong Law of Large Numbers implies that $\overline{\psi}_i$ converges in probability to ψ_i. From the Lindeberg-Levy Central Limit Theorem, we obtain the result that $\sqrt{N}(\overline{\psi_i} - \psi_i)$ converges in distribution to $N(0,\sigma_i^2)$. Finally, from the Cramer-Wald Theorem, it can be shown that $\sqrt{N}(\overline{\Psi} - \Psi)$ converges to a multivariate normal distribution, $N(\underline{0},\Omega)$. Q.E.D.

Theorem 3 allows us to analyze the sampling distribution of the estimated decomposed ILC ordinates. Applying Rao's (1965) theorem on the limiting distribution of differentiable functions of random variables, the limiting distribution of $\hat{\Phi}$ is also multivariate normal. We summarize this result in the following theorem.

THEOREM 4. *Under the conditions of Theorem 3, the vector of estimated decomposed ILC ordinates* $\underset{1x(mK)}{\hat{\Phi}} = (\hat{\Phi}_1, \hat{\Phi}_2, ..., \hat{\Phi}_{mK})' = \big((\hat{\phi}_{1,(1)}^h, \ldots, \hat{\phi}_{m,(1)}^h), (\hat{\phi}_{1,(2)}^h, \ldots, \hat{\phi}_{m,(2)}^h), \ldots, (\hat{\phi}_{1,(K)}^h$

$,\ldots,\hat{\phi}^h_{m,(K)}))'$ *is asymptotically normal in that* $\sqrt{N}(\hat{\Phi}-\Phi)$ *has a limiting Km -variate normal distribution with mean zero and covariance matrix* $V = J\Omega J' = (v_{ij})$*, where* Ω *is defined in Theorem 3 and J is defined as* $\underset{(2Km)x(2Km+k)}{J} = \left[\delta\overline{\Phi}_j / \delta\overline{\Psi}_j\right]_{\overline{\Psi}=\Psi}$.

Then, the covariance estimate of the *k*-th and *l*-th estimated decomposed ILC ordinates, $\hat{\phi}^h_{j,(k)}$ and $\hat{\phi}^h_{j,(l)}$, can then be determined as follows:

$$\begin{aligned}\overline{Cov}(\hat{\phi}^h_{j,(k)},\hat{\phi}^h_{j,(l)}) = a_k a_l &\left[\overline{(xG^x_k I^x_{\tau_j})^h (xG^x_l I^x_{\tau_j})^h} - \left(\overline{(xG^x_k I^x_{\tau_j})^h}\right)\left(\overline{(xG^x_l I^x_{\tau_j})^h}\right)\right] + \\ c_{jk}c_{jl} &\left[\overline{(xG^x_k)^h (xG^x_l)^h} - \left(\overline{(xG^x_k)^h}\right)\left(\overline{(xG^x_l)^h}\right)\right] + \\ a_k c_{jk} &\left[\overline{(xG^x_k I^x_{\tau_j})^h (xG^x_k)^h} - \left(\overline{(xG^x_k I^x_{\tau_j})^h}\right)\left(\overline{(xG^x_k)^h}\right)\right] + \\ a_l c_{jl} &\left[\overline{(xG^x_l I^x_{\tau_j})^h (xG^x_l)^h} - \left(\overline{(xG^x_l I^x_{\tau_j})^h}\right)\left(\overline{(xG^x_l)^h}\right)\right]\end{aligned}$$

where $a_k = \left(\overline{(xG^x_k)^h}\right)^{-1}$, $a_l = \left(\overline{(xG^x_l)^h}\right)^{-1}$, $c_{jk} = -\left(\overline{(xG^x_k I^x_{\tau_j})^h}\right)\left(\overline{(xG^x_k)^h}\right)^{-2}$, and $c_{jl} = -\left(\overline{(xG^x_l I^x_{\tau_j})^h}\right)\left(\overline{(xG^x_l)^h}\right)^{-2}$ for $j = 1,2,\ldots,m$.

One can perform a goodness-of-fit test for a *marginal* change in the subgroup moments. That is, one can test for differences in the decomposed moments for subgroup 1 (e.g., $\underset{mx1}{\hat{\psi}} = (\hat{L}^{(1)}_1 - \hat{C}^{(1)}_1,\ldots,\hat{L}^{(1)}_m - \hat{C}^{(1)}_m)'$). Under the null hypothesis that $H_0:\ \psi = \underline{0}$, an appropriate test statistic is

(8) $$c = N\hat{\psi}'\hat{\Theta}^{-1}\hat{\psi},$$

where $$\hat{\Theta} = \rho\hat{V}\rho',$$

and $$\rho = \begin{bmatrix}\begin{bmatrix}1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & \cdots & 0\end{bmatrix} & \begin{bmatrix}-1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & \cdots & 0\end{bmatrix}\end{bmatrix}_{m\ \mathrm{x}\ 2Km}.$$

From Theorem 3, the c-statistic is also asymptotically distributed as a (central) chi-squared variate with *m*-degrees of freedom.

Alternatively, one could test a joint hypothesis such that $\{H_{0j} : \psi_j = L_j^{(1)} - C_j^{(1)} = 0 \mid j = 1,...,m\}$. Then the appropriate test statistics are

(9) $$Z_j = \sqrt{N}\,\frac{\hat{\psi}_j}{\sqrt{\hat{\Theta}_{jj}/N}}, \text{ for } j = 1, 2, ..., m.$$

Let $Z^* = \max_{1\le j\le m} |Z_j|$ be the largest absolute value of the test statistics. We then apply the Sidak (1967) probability inequality and the results in Hochberg (1974) and Richmond (1982) to control the size of the multiple sub-hypothesis tests.

THEOREM 5. *Let* $Z = (Z_1,...,Z_m)'$ *be a vector of m test statistics corresponding to* (9). *From Theorem 4, the distribution of vector Z converges asymptotically to an m-variate normal distribution. Under the null hypothesis, the confidence interval of at least* $100(1-\alpha)$ *percent for the extreme statistic,* $Z^*(\tau)$, *can be defined as:*

(10) $$Z^*(\tau) \pm SMM(\alpha; m; \infty),$$

where $SMM(\alpha; m; \infty)$ *is the asymptotic critical value of the* α*-point of the Studentized Maximum Modulus (SMM) distribution (Stoline and Ury 1979) with parameter m and* ∞ *degrees of freedom.*

Further, let $Z^{*+} = \max_{1\le j\le m} Z_j$, and $Z^{*-} = \min_{1\le j\le m} Z_j$. The asymptotic joint confidence interval of at least $100(1-\alpha)$ percent is:

(11) $$-SMM(\alpha; m; \infty) \le Z^{*-} ... \le Z^{*+} \le SMM(\alpha; m; \infty).$$

We emphasize that test statistic c in (8) illustrates only one possibility, and that the results of Theorem 3 and the SMM approach for controlling the joint test size can be applied to a wide range of hypothesis tests for subgroup income distribution comparisons. We also note that the proposed methodology employs a finite-target testing approach, so it may have low power in detecting tail inequality for fat-tail distributions. Thus, the power of the tests is an issue for further research.

To illustrate our statistical inference procedures, we apply them to incomes of whites and nonwhites in the CPS. We estimate ILCs at deciles for these groups and test whether they differ from the 45-degree line in a particular year and from each other across years, using data from the CPS in 1977, 1987, 1992, 1997, and 2002. We avoid comparisons during 1993-95, when the CPS made substantial changes in the top-coding of incomes in the public-use sample (Burkhauser, et. al., 2004). Our sample includes only primary families (excluding single-person families and unrelated individuals). We correct for inflation to allow pooling of incomes across time, but we make no adjustments for the size and composition of the family.

Table 1 presents the mean incomes of whites and nonwhites, adjusted for inflation, over the years we consider. Before the change in top-coding (1976-91), the mean incomes for both population subgroups increased slowly, 5.45 percent for whites and 3.86 percent for nonwhites. After the change in top-coding (1996-2001), the mean incomes increased much more rapidly, 13.54 percent for whites and 11.37 for nonwhites. By construction, the ILCs are sensitive to changes in both the level and dispersion of the distributions by population subgroup.

Table 2 presents statistical tests for white-nonwhite ILCs, based on population shares ($h = 0$) in 1976 and 1991, before the change in top-coding for the CPS public-use samples. Column (1) shows the target incomes, which are decile order statistics for the distribution of incomes pooled across subgroup and time. Columns (2) and (3) report the estimated population shares in 1976 for whites and nonwhites at or below each target income, with standard errors in parentheses. Column (4) gives differences in population shares by race. Columns (5)–(7) give the corresponding information for 1991. Column (8) reports the "difference in differences" over time, while column (9) provides the test statistics, Z_j, j=1,2,…,10, from equation (9) in section 2. We also give the chi-squared test statistic c from equation (8) in section 2 in the bottom row of Table 2.

An inspection of columns (4) and (7) of Table 2 reveals that the estimated differences between the 45-degree line and the 1976 and 1991 ILCs are large relative to their standard errors. The chi-squared statistics in both columns are highly significant as well. Both results indicate that whites have an economic advantage over nonwhites for these years. Column (8) alerts us to a possible crossing of ILCs for 1976 and 1991, because the difference in differences yields both negative and positive signs. The SMM test statistics in column (9), however, do not support a crossing, because no positive test statistic is significant at the ten-percent level. Hence, we find that the white-nonwhite ILC (for population shares) shifted *away from* the 45-degree line during 1976-91, creating a widening advantage for whites over nonwhites during the period.

To explore why the advantage of whites over nonwhites (in population shares) changed over time, consider the rows of Table 2 in more detail. Recall (Table 1) that mean incomes rose ─ albeit slowly ─ during 1976-91 and grew slightly faster for whites than nonwhites. Nevertheless, Table 2 reveals that population shares at or below fixed target incomes *rose* in the bottom four deciles for whites and nonwhites, which mean that incomes were falling at the bottom of the distribution. Population shares declined in the upper six deciles for whites and nonwhites, indicating rising incomes (as reflected in the rising means). These patterns show a widening dispersion of incomes in both subgroups.

The changes in both the level and dispersion of incomes in each subgroup influence the comparisons of ILCs over time, because they affect the population shares at or below fixed incomes.

Table 3 shows the statistical tests for the white-nonwhite ILCs, based on income shares (h = 1) instead of population shares in 1976 and 1991. Once again, the nonwhite-white differences in columns (4) and (7) are large relative to their standard errors and the chi-squared statistics in these columns are highly significant, which implies an advantage for whites over nonwhites using income shares. In column (8), a few positive "difference in differences" are statistically significant, but the negative one (in the top decile) is not, so the ILC for income shares *converged* toward the 45-degree line, unlike the population-share ILC in Table 3. This finding is reinforced by the chi-squared statistic in column (8), which is statistically significant.[4]

Table 4 gives a summary of the ILC comparisons. We show comparisons for population shares (h = 0) and income shares (h = 1), and for different periods. The first row gives the results for a period (1976-91) in which the ILC based on population shares is diverging, while the ILC based on income shares is converging. The second row gives the results for a period (1986-91) that illustrates no significant differences between ILCs, using population shares or income shares. The last row presents a period (1996-2001) in which we find a statistically significant crossing using population shares, but converging ILCs using income shares. In all cases except the crossing, the chi-squared statistic and

[4] Reardon, *et. al.* (2006), who study urban income segregation, plot the inverse of our decomposed Lorenz ordinates (our Table 3; their Figures 1 and 2) for various census tracts in San Francisco and Detroit. In contrast to the ILC, whose analog would be to compare two Census tracts, they compare the individual Census tracts to the city average. Their Figures 3 and 4 are similar to a plot of our white-nonwhite differences (Table 3, column 3) for all Census tracts. Given the large number of tracts in each city, they follow an index number approach to presenting their findings. With fewer comparisons to make, we follow the dominance approach, which avoids the ambiguities arising from a multiplicity of index numbers.

the dominance comparisons yield identical conclusions, but the chi-squared statistic is misleading in the case of a crossing.

As Table 4 shows, the techniques presented here allow researchers to identify movements toward greater interdistributional equality or inequality between two groups by testing for convergence or divergence in ILCs over time. Once such movements have been identified, it would be natural to seek explanations for them, but that is beyond the scope of this paper, and must be left for future research.

Conclusions

We have demonstrated that the ILCs of Butler and McDonald (1987) offer a convenient visual representation of convergence or divergence between distributions of income over time, and we have proposed methods for testing whether the movements are statistically significant. These tests apply to ILCs based on population shares and income shares – or any incomplete moment of the distributions – and impose no prior restrictions on the functional form of the underlying distribution.

We have illustrated these methods with applications to LIS data on income distributions in Spain, Italy, and Germany in the 1990s, and to CPS data on distributions of income for whites and nonwhites from the 1970s through 2001. We find convergence between Spain and Italy, and Spain and Germany, but not between Italy and Germany. In comparisons of income distributions by race in the United States, the findings depend on the time frame and on the choice of population or income shares for constructing the ILCs.

Figure 2
Convergence between Spain and Italy?
ILCs for 1990 and 2000

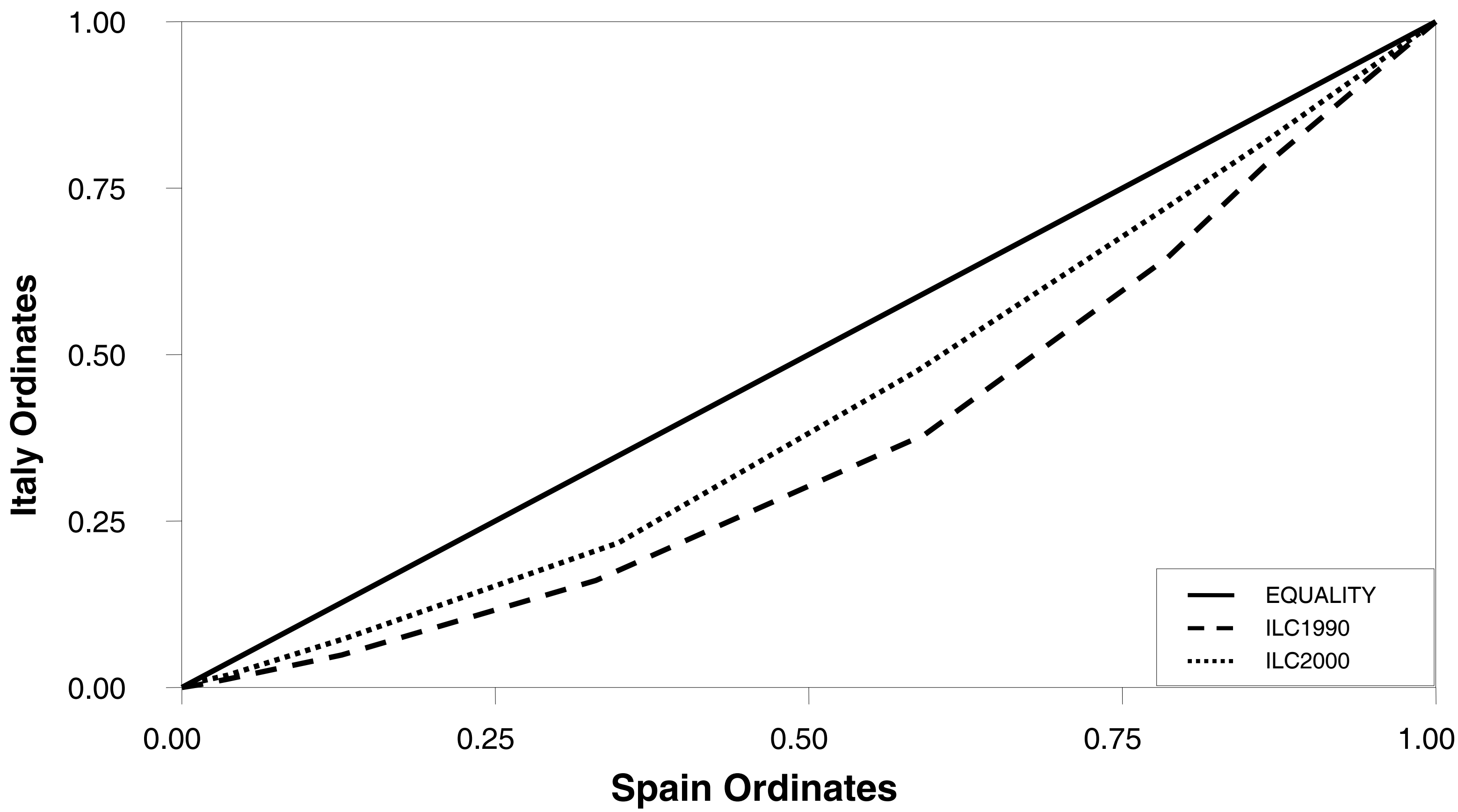

Figure 3
Convergence between Spain and Germany?
ILCs for 1990 and 2000

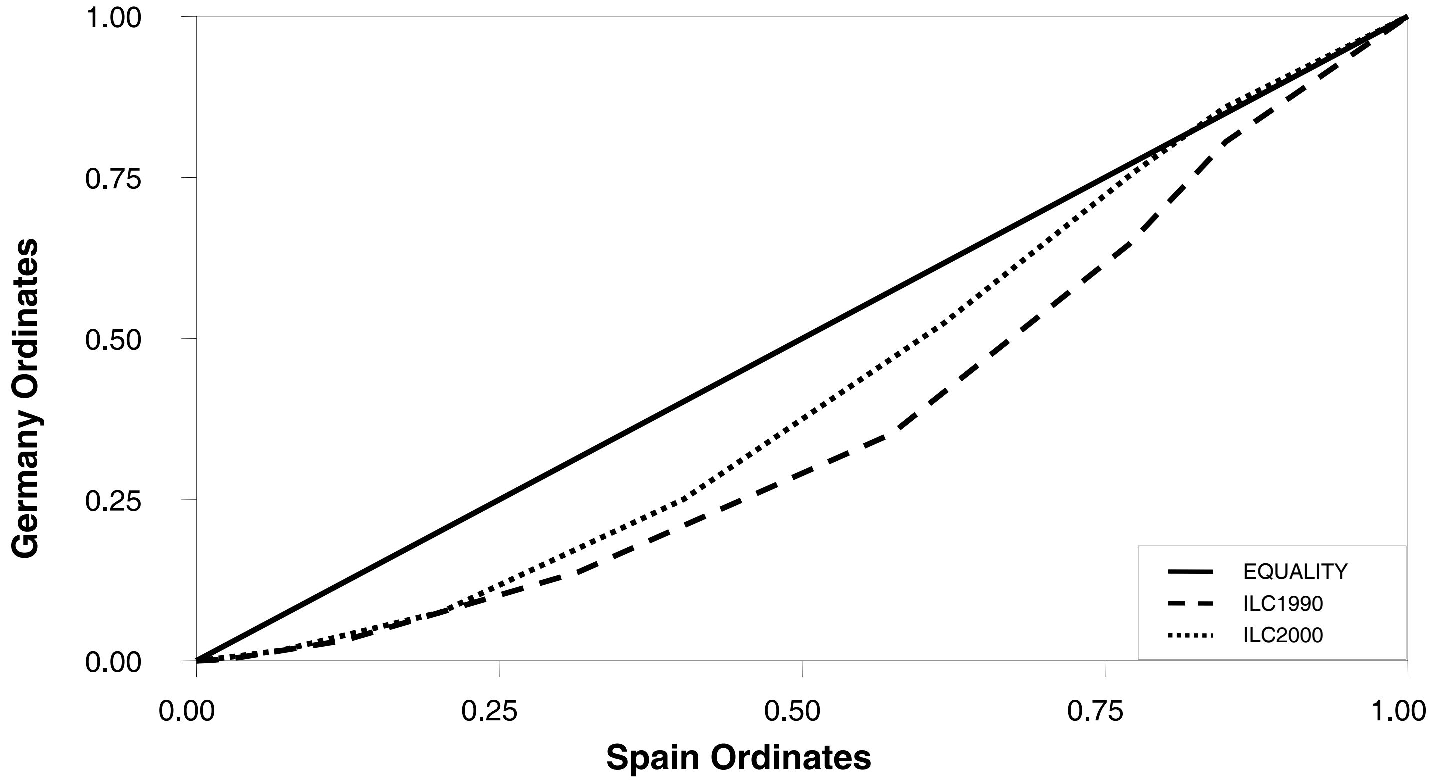

Figure 4
Convergence between Italy and Germany?
ILCs for 1990 and 2000

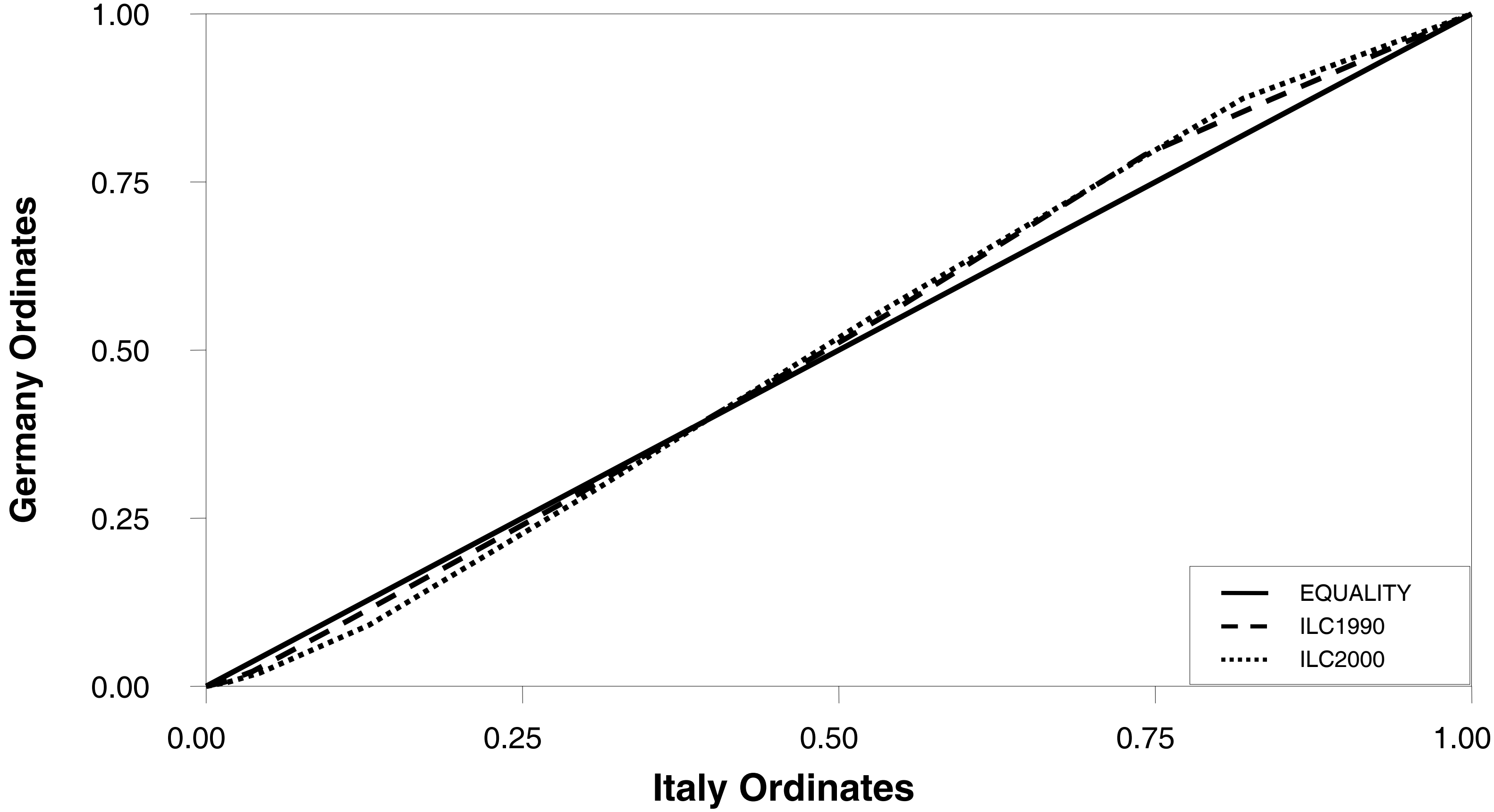

Table 1
Mean Incomes for Whites and Nonwhites

Year	White	Nonwhite
1976	53,393 (183.0)	35,483 (426.0)
1986	57,193 (217.0)	37,736 (497.0)
1991	56,303 (215.0)	36,853 (502.0)
1996	61,646 (335.0)	40,886 (776.0)
2001	69,995 (301.0)	45,535 (541.0)

Note: All incomes are expressed in 2001 dollars. The numbers in parentheses are standard errors.

Table 2
Interdistributional Lorenz Ordinates (Population Shares) for Nonwhite vs White: 1976 and 1991

Target Income (1)	1976 White Share (2)	1976 Nonwhite Share (3)	1976 Nonwhite-White Difference (4)=(3)-(2)	1991 White Share (5)	1991 Nonwhite Share (6)	1991 Nonwhite-White Difference (7)=(6)-(5)	Converging ILCs? Difference in Differences 8=(4)-(7)	Converging ILCs? Test Statistic (9)
14,728	0.0768 (0.0014)	0.2352 (0.0070)	0.1584 (0.0071)	0.0945 (0.0015)	0.2811 (0.0072)	0.1865 (0.0073)	-0.0281 (0.0102)	-2.76
23,058	0.1767 (0.0020)	0.4083 (0.0081)	0.2317 (0.0083)	0.1901 (0.0021)	0.4145 (0.0078)	0.2244 (0.0081)	-0.0073 (0.0112)	0.63
31,025	0.2767 (0.0023)	0.5277 (0.0082)	0.2510 (0.0085)	0.2926 (0.0024)	0.5290 (0.0079)	0.2364 (0.0083)	0.0146 (0.0112)	1.23
39,042	0.3878 (0.0025)	0.6290 (0.0079)	0.2412 (0.0083)	0.4000 (0.0026)	0.6313 (0.0077)	0.2314 (0.0081)	0.0099 (0.0112)	0.85
47,808	0.5038 (0.0026)	0.7355 (0.0072)	0.2317 (0.0077)	0.4995 (0.0026)	0.7139 (0.0072)	0.2143 (0.0077)	0.0173 (0.0108)	1.59
56,798	0.6245 (0.0025)	0.8191 (0.0063)	0.1946 (0.0068)	0.6010 (0.0026)	0.7931 (0.0065)	0.1921 (0.0069)	0.0025 (0.0097)	0.26
67,814	0.7390 (0.0023)	0.8901 (0.0051)	0.1510 0.0000	0.7006 (0.0024)	0.8592 (0.0055)	0.1586 (0.0060)	-0.0076 (0.0082)	-0.92
82,377	0.8455 (0.0019)	0.9438 (0.0038)	0.0983 (0.0042)	0.8005 (0.0021)	0.9137 (0.0045)	0.1131 0.0049	-0.0148 (0.0065)	-2.28
108,398	0.9353 (0.0013)	0.9847 (0.0020)	0.0494 (0.0024)	0.9031 (0.0016)	0.9648 (0.0029)	0.0617 (0.0033)	-0.0123 (0.0041)	-3.02
Chi-Square Statistic			1456.7			1267.2	40.0	

Note: The target incomes are the decile order statistics for the distribution pooled across years and population subgroups expressed in 2001 dollars. numbers in parenthesis are standard errors. The test statistic for converging ILCs is a student maximum modulus (SMM). The critical values of the smm for the 5-percent and 10-percent levels are 2.76 and 2.52, respectively.

Table 3
Interdistributional Lorenz Ordinates (Income Shares) for Nonwhite vs White: 1976 and 1991

	1976			1991			Converging ILCs?	
Target Income (1)	White Share (2)	Nonwhite Share (3)	Nonwhite-White Difference (4)=(3)-(2)	White Share (5)	Nonwhite Share (6)	Nonwhite-White Difference (7)=(6)-(5)	Difference in Differences 8=(4)-(7)	Test Statistic (9)
14,728	0.0139 (0.0003)	0.0634 (0.0025)	0.0495 (0.0025)	0.0150 (0.0003)	0.0608 (0.0023)	0.0458 (0.0023)	0.0037 (0.0034)	1.09
23,058	0.0489 (0.0007)	0.1538 (0.0047)	0.1049 (0.0048)	0.0470 (0.0006)	0.1273 (0.0041)	0.0802 (0.0041)	0.0246 (0.0063)	3.91
31,025	0.0992 (0.0011)	0.2446 (0.0066)	0.1454 (0.0067)	0.0955 (0.0011)	0.2100 (0.0060)	0.1146 (0.0061)	0.0309 (0.0090)	3.42
39,042	0.1717 (0.0016)	0.3416 (0.0083)	0.1744 (0.0085)	0.1600 (0.0016)	0.3029 (0.0028)	0.1429 (0.0029)	0.0315 (0.0116)	2.71
47,808	0.2613 (0.0022)	0.4728 (0.0099)	0.2115 (0.0101)	0.2353 (0.0021)	0.4013 (0.0093)	0.1660 (0.0095)	0.0456 (0.0139)	3.27
56,798	0.3762 (0.0027)	0.5917 (0.0108)	0.2156 (0.0111)	0.3268 (0.0026)	0.5105 (0.0106)	0.1837 (0.0109)	0.0319 (0.0156)	2.04
67,814	0.5058 (0.0031)	0.7135 (0.0110)	0.2077 (0.0114)	0.4335 (0.0030)	0.6231 (0.0114)	0.1897 (0.0118)	0.0180 (0.0164)	1.10
82,377	0.6549 (0.0034)	0.8232 (0.0103)	0.1683 (0.0108)	0.3649 (0.0034)	0.7308 (0.0115)	0.1659 (0.0120)	0.0024 (0.0162)	0.15
108,398	0.8140 (0.0032)	0.8365 (0.0076)	0.1225 (0.0083)	0.7341 (0.0036)	0.8628 (0.0103)	0.1288 (0.0109)	0.0063 (0.0127)	-0.45
Chi-Square Statistic			746.8			564.2	27.0	

Note: The target incomes are the decile order statistics for the distribution pooled across years and population subgroups expressed in 2001 dollars. numbers in parenthesis are standard errors. The test statistic for converging ILCs is a student maximum modulus (SMM). The critical values of the smm for the 5-percent and 10-percent levels are 2.76 and 2.52, respectively.

Table 4
Summary of ILC Comparisons

Period	**Population Shares (h = 0)**	**Income Shares (h = 1)**
1976-1991	Diverging (40.0)	Converging (27.0)
1986-1991	No Difference (5.9)	No Difference (3.4)
1996-2001	Crossing (48.4)	Converging (45.1)

Note: The numbers in parentheses are chi-squared statistics. The critical value for the chi-squared statistic is 16.9.

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