Fundamental Non-Convexity and Externalities: A Differentiable Approach

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Abstract

It is well known that externalities can cause fundamental nonconvexity problems in the production sets (Baumol 1972, Starett 1972). We use the differentiable approach to establish existence without requiring aggregate convexity in consumption nor production. Our model also allows price dependent externalities and individual preferences that are not convex in externalities.

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1 Introduction

Externalities prevail in the real world, yet they are difficult to deal with in general equilibrium models. Baumol (1972) first points out that the aggregate production possibility set of the polluter's activity and the pollutee's activity may present itself a nonconvex set when the external damages are strong. For example, when a laundry (pollutee) and a steel mill (polluter) locate side by side, the production frontier becomes L-shaped with only the production of either of the two commodities possible. Even though individual production and consumption sets are convex, externalities create nonconvexity in the aggregate, which presents a problem for the conventional convex analysis approach to finite economies. Moreover, the price hyperplane needs to separate each firm's and

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consumer's production sets independently, yet with externalities, these sets are not independent. Another type of fundamental nonconvexity is pointed out by Starrett (1972). When a positive price for pollution rights is determined in the Arrovian externalities market, the pollutee may want to sell an infinite amount of rights. Boyd and Conley (1997) argue that this type of nonconvexity can be resolved by specifying an endowment bound for pollution rights. On the other hand, the Baumol type of nonconvexity still persists.

Our paper presents a differentiable approach to externalities where convexity in the aggregate production or consumption is not required. Externalities are allowed to influence production and consumption in arbitrary ways. As long as consumer preferences and firms' production sets are convex in own activities, being demand or net output, for fixed levels of externalities, a competitive equilibrium exists under standard assumptions. Our approach studies equilibrium of an economy as the intersection of manifolds, in line with Mas-Colell (1985), Balasko (1988), and Geanakoplos and Shafer (1990). A nonempty intersection obtains if these manifolds are transversal and the fixed point mapping needs not to be convex valued.

The following authors address issues of externalities in competitive equilibrium. Bonnisseau and del Mercato (2010) study externalities when consumer have consumption constraints. Kung (2008) presents a public goods model with externalities in consumption (but not in production). Noguchi and Zame (2006) use a continuous model of a distribution of consumptions on indivisible goods, convexity is not required though. Cornet and Topuzu (2005) study a two-period temporary equilibrium model as a reduced Walrasian economy with price dependency externalities. Balder (2003) demonstrates that equilibrium exists if the externalities enter into preferences of each individual in the same way (which seems to exclude local externalities, externalities that diminish with distance, and externalities that have directional effects). Greenberg and Shitovitz's (1979) approach models an abstract economy that allows price dependency and consumption externalities (though there is aggregate production but no individual firms.) In contrast to the literature, our model allows for production set and individual preferences that are not convex in externalities, and general externalities that firms and consumers experiences in unrestricted ways.

The problem of convexity associated with externalities is illustrated in Figure 1 (Baumol 1972, p.317). As the degree of external damages are getting stronger, the production frontier of x_1 and x_2 moves from C_1, C_2, ..., to C_{S+1} with AOB as its limit.

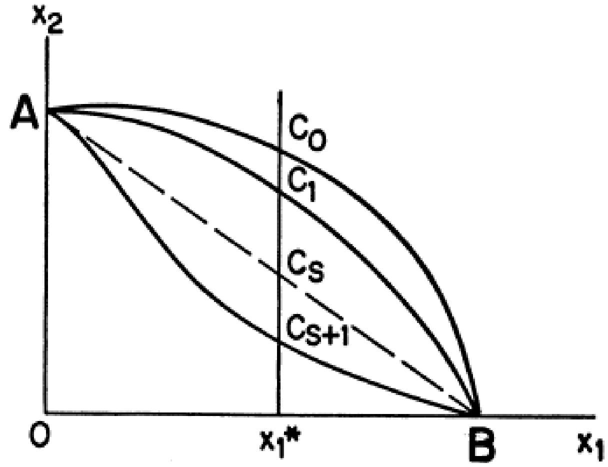

Figure 1: Nonconvexity in the joint production set.

How this convexity problem can be handled by the differentiable approach is illustrated in Figure 2 (Geanakoplos and Shafer 1990, p.71, $\Phi(p) - \hat{z}(p) = 0$ being the solution). The middle panel has a convex-valued $\Phi(p)$ map and admits a fixed point. The left panel has a discontinuous $\Phi(p)$ map and there is no fixed point. The right panel has a continuous map which is not convex-valued, yet it admits a fixed point. The right panel shows $\Phi(p)$ as a differentiable manifold, where a fixed point obtains without convexity.

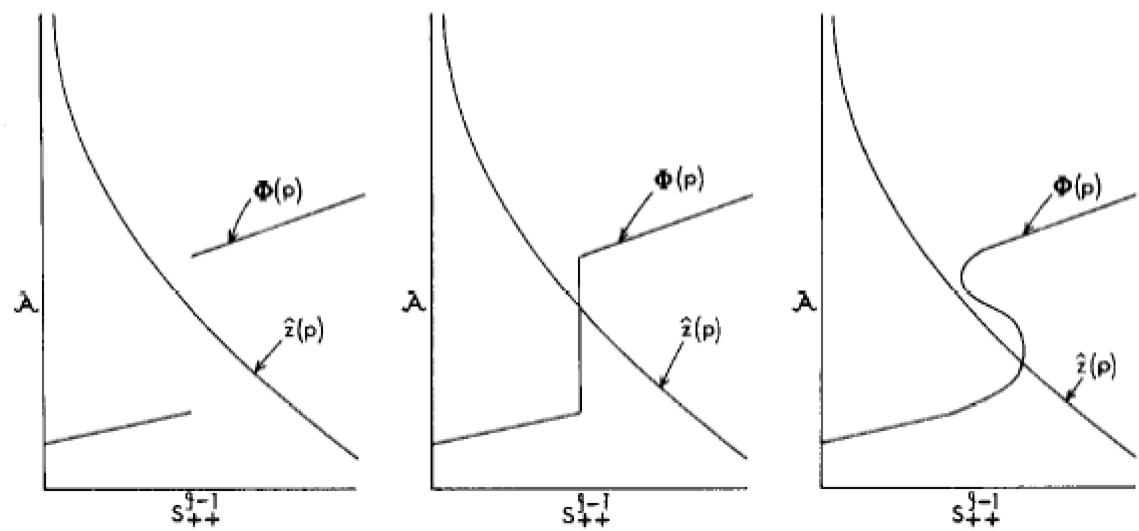

Figure 2: Fixed point in a nonconvex-valued map.

We extend this differentiable approach to include production and externalities. Section 2 introduces the model and main result. Section 3 concludes.

2 The Production Economy

There are N private goods, I consumers, and J firms. The prices of private goods are denoted by $p \in S^N$ where $S^N = \left\{p \in \Re^N_{++} \mid \sum_{n=1}^{N} p_n = 1\right\}$ is the interior of the $(N-1)$-dimensional simplex.[1] Let $x_i \in \Re^N_{++}$ denote the consumption bundle of consumer i, and $y_j \in \Re^N$ denote net output of firm j. The activities of all consumer and firms enter into the utility functions of every consumer and the production technology of every firm. Each of consumer i and firm j is influenced by a profile of externalities including equilibrium prices. Let $T_i = \left((x_h)^I_{h=1,h\neq i}, (y_j)^J_{j=1}\right)$ for consumer i, and $T_j = \left((x_i)^I_{i=1}, (y_h)^J_{h=1,h\neq j}\right)$ for firm j. All external activities are recorded as positive amounts. This model keeps track of the amount of the original activities such as the consumption of cigarettes, instead of the external by-products of these activities such as the amount of second-hand smoke.

The production technology of firm j is represented by a C^2 transformation function $f_j(y_j, T_j, p) : \Re^{IN}_{++} \times \Re^{JN} \times S^N \to \Re$, which follows standard assumptions: f_j is *differentiably strictly decreasing* in y_j, i.e., $D_{y_j} f_j \ll 0$. f_j is *differentiably strictly quasiconcave* in y_j, i.e., if $D_{y_j} f_j v = 0$, then $v D^2_{y_j} f_j v < 0$ for all $v \in \Re^N \setminus \{0\}$.

Firm j taking prices and externalities as given maximizes profit py_j over $y_j \in \Re^N$ subject to $f_j(y_j, T_j, p) = 0$. With $\nu_j \in \Re$ as the multiplier, the first order conditions are

$$p - \nu_j D_{y_j} f_j(y_j, T_j, p) = 0,$$
$$f_j(y_j, T_j, p) = 0.$$

Each consumer i is endowed with private goods $e_i \in \Re^N_{++}$ and a share $s_{ij} \in [0,1]$ of firm j. Preferences of consumer i are represented by a C^2 utility function $u_i(x_i, T_i, p) : \Re^{IN}_{++} \times \Re^{JN} \times S^N \to \Re$, which follows standard assumptions: u_i is *differentiably strictly increasing* in x_i, i.e., $D_{x_i} u_i \gg 0$. u_i is *differentiably strictly quasiconcave* in x_i, i.e., if $D_{x_i} u_i v = 0$ then $v D^2_{x_i} u_i v < 0$ for all $v \in \Re^N \setminus \{0\}$. u_i satisfies the *boundary condition*[2]: for all T_i such that for any bundle $x'_i \in \Re^N_{++}$, the upper contour set $\left\{x_i \in \Re^N_{++} \mid u_i(x_i, T_i, p) \geq u_i(x'_i, T_i, p)\right\}$ is closed in $\Re^N_{++}$.

Consumer i maximizes utility $u_i(x_i, T_i, p)$ over $x_i \in \Re^N_{++}$ subject to budget $p(x_i - e_i) -$

[1] We can safely exclude zero prices because of assumed quasiconcavity.

[2] This guarantees an interior solution with positive demands for all good (Mas-Colell 1985).

$\sum_{j=1}^{J} s_{ij} p y_j = 0$. The first order conditions are

$$D_{x_i} u_i \left(x_i; T_i, p\right) - \lambda_i p = 0,$$
$$p\left(x_i - e_i\right) - \sum_{j=1}^{J} s_{ij} p y_j = 0,$$

with $\lambda_i \in \Re$ being the multiplier. The markets clear with

$$\sum_{i=1}^{I} \left(x_i - e_i\right) - \sum_{j=1}^{J} y_j = 0.$$

Definition 1. An *equilibrium* of the benchmark economy (e, s) is a list $\left(\left(x_i\right)_{i=1}^{I}, \left(\lambda_i\right)_{i=1}^{I}, \left(y_j\right)_{j=1}^{J}, \left(\nu_j\right)_{j=1}^{J}, p\right)$ that satisfies the following C^1 equations, where $\left(x_i\right)_{i=1}^{I} \in \Re_{++}^{IN}$ are consumption bundles, $\left(y_j\right)_{j=1}^{J} \in \Re^{JN}$ are production plans, $p \in S^N$ is the price vector, $\left(\lambda_i\right)_{i=1}^{I} \in \Re^{I}$ and $\left(\nu_j\right)_{j=1}^{J} \in \Re^{J}$ are multipliers.

$$D_{x_i} u_i \left(x_i, T_i, p\right) - \lambda_i p = 0, \forall i,$$
$$p\left(x_i - e_i\right) - \sum_{j=1}^{J} s_{ij} p y_j = 0, \forall i,$$
$$p - \nu_j D_{y_j} f_j \left(y_j, T_i, p\right) = 0, \forall j,$$
$$f_j \left(y_j, T_i, p\right) = 0, \forall j,$$
$$\sum_{i=1}^{I} \left(x_i - e_i\right) - \sum_{j=1}^{J} y_j = 0.$$

Perturbing the economy

Take ε small enough so that it does not alter the properties of u_i and f_j assumed above. We perturb the utility function with $\alpha_i \in \Re_{+}^{N}$.

$$u_i \left(x_i, T_i, p\right) + \varepsilon \alpha_i x_i.$$

Firm specific parameters $\beta_j \in \Re^N$ and $\gamma_j \in \Re$ (let $\gamma = \left(\gamma_j\right)_{j=1}^{J}$) perturb around transformation function f_j.

$$f_j \left(y_j, T_j, p\right) + \varepsilon \left(\beta_j y_j + \gamma_j\right).$$

Let $s_{-1} = \left(s_{i1}\right)_{i=1, i \neq 1}^{I}$; it is the profile of all consumers' shares of firm $j = 1$ except for $i = 1$. We will use the augmented parameter space $\theta = \left(\left(\alpha_i\right)_{i=1}^{I}, s_{-1}, \left(\beta_j\right)_{j=1}^{J}, \gamma, e_1\right) \in \Theta \subset \Re_{+}^{IN} \times [0,1]^{I-1} \times \Re^{JN} \times \Re^{J} \times \Re_{++}^{N}$. The benchmark model is parameterized at

$(0, s_{-1}, 0, 0, e_1)$. These parameters perturb the system orthogonally so that its Jacobian matrix has full rank, which provides enough independent directions for it to be transversal. This technique can disentangle the interdependency generated by externalities. .

Definition 2. An *equilibrium* of the economy θ in the augmented parameter space Θ is a list $\left((x_i)_{i=1}^I, (\lambda_i)_{i=1}^I, (y_j)_{j=1}^J, (\nu_j)_{j=1}^J, (p)\right) \in \Xi = \Re_{++}^{IN} \times \Re^I \times \Re^{JN} \times \Re^J \times S^N$, that satisfy the following conditions:

$$
\begin{aligned}
D_{x_i} u_i (x_i, T_i, p) + \varepsilon \alpha_i - \lambda_i p = 0, \forall i, \\
p (x_i - e_i) - \sum_{j=1}^J s_{ij} p y_j = 0, \forall i \neq 1. \\
p - \nu_j \left(D_{y_j} f_j (y_j, T_j, p) + \varepsilon \beta_j\right) = 0, \forall j, \\
f_j (y_j, T_j, p) + \varepsilon \left(\beta_j y_j + \gamma_j\right) = 0, \forall j, \\
\sum_{i=1}^I (x_i - e_i) - \sum_{j=1}^J y_j = 0.
\end{aligned}
\tag{1}
$$

The budget constraint of $i = 1$ satisfies automatically by Walras' Law. Denote the left-hand side of system (1) as a C^1 map $\phi : \Xi \times \Theta \to \Re^{IN+I+JN+J+N-1}$. Let $\chi \in \Xi$ denote an element of Ξ.

Theorem 1. *Equilibrium exists for every economy $\theta \in \Theta$.*

Proof. First, a simplified seed economy without externalities is defined as follows. Let $\hat{u}(x_i) = \sum_{i=1}^I \ln x_{in}/N$; consumers have preferences $\hat{u}(x_i) + \varepsilon \alpha_i$. Firm 1 has linear production technology $\beta_1 y_1 + \gamma_1 = 0$. Take an differentiably strictly decreasing and quasiconcave function $\hat{f}(y_j)$, the transformation functions for other firms $j \neq 1$ are $\hat{f}(y_j) + \varepsilon \left(\beta_j y_j + \gamma_j\right)$. Thus, the following C^1 map η,

$$
\eta (\chi, \theta) = \begin{pmatrix}
D_{x_i} \hat{u} (x_i) + \varepsilon \alpha_i - \lambda_i p, \forall i \\
p (x_i - e_i), \forall i \neq 1 \\
p - \nu_1 \beta_1 \\
p - \nu_j \left(D_{y_j} \hat{f}_j (y_j) + \varepsilon \beta_j\right), \forall j \neq 1 \\
\beta_1 y_1 + \gamma_1 \\
\hat{f} (y_j) + \varepsilon \left(\beta_j y_j + \gamma_j\right), \forall j \neq 1 \\
\sum_{i=1}^I (x_i - e_i) - \sum_{j=1}^J y_j
\end{pmatrix}
$$

defines the equilibrium of the seed economy at $\eta(\chi, \theta) = 0$. There is a unique solution χ' as follows: We can solve prices as $p' = \nu_1' \beta_1$, then $\nu_1' = 1/\sum_{h=1}^N \beta_{1h}$. Prices p' then uniquely determine the production plan y_j', multiplier ν_j' of firm $j \neq 1$, consumer i's bundle x_i', and multiplier λ_i' due to strict quasiconcavity of the transformation functions and utility functions. Finally, $y_1' = \sum_{i=1}^I (x_i' - e_i) - \sum_{j=1}^J y_j'$.

This seed economy $\eta(\chi, \theta)$ will be deformed continuously into ϕ via a homotopy while its topological properties are preserved. Define a homotopy $\Phi : \Xi \times [0,1] \times \Theta \to \Re^{IN+I+JN+J+N-1}$ where $\Phi(\chi, 0, \theta) = \eta(\chi, \theta)$ and $\Phi(\chi, 1, \theta) = \phi(\chi, \theta)$.

$$\Phi(\chi, \rho, \theta) = \begin{pmatrix} \rho D_{x_i} u_i (x_i, T_i, p) + (1-\rho) D_{x_i} \hat{u}(x_i) + \varepsilon \alpha_i - \lambda_i p, \forall i \\ p(x_i - e_i) - \rho \sum_{j=1}^J s_{ij} p y_j, \forall i \neq 1 \\ p - \rho \nu_1 (D_{y_1} f_1 (y_1, T_1, p) + \varepsilon \beta_1) - (1-\rho) v_1 \beta_1 \\ p - \nu_j \left(\rho D_{y_j} f_j (y_j, T_j, p) + (1-\rho) D_{y_j} \hat{f}_j (y_j) + \varepsilon \beta_j \right), \forall j \neq 1 \\ \rho f_1 (y_1, T_1, p) + (\rho \varepsilon + 1 - \rho)(\beta_1 y_1 + \gamma_1) \\ \rho f_j (y_j, T_j, p) + (1-\rho) \hat{f}(y_j) + \varepsilon \left(\beta_j y_j + \gamma_j \right), \forall j \neq 1 \\ \sum_{i=1}^I (x_i - e_i) - \sum_{j=1}^J y_j \end{pmatrix}.$$

The following lemma shows that the preimage of the homotopy is closed.

Lemma 1. $\Phi^{-1}(0) = \{(\chi, \rho, \theta) \in \Xi \times [0,1] \times \Theta \mid \Phi(\chi, \rho, \theta) = 0\}$ *is closed in* $\Re^{2(IN+I+JN+J+N-1)+1} \times [0,1] \times \Theta$.

Proof. Take a sequence $(\chi_k, \rho_k, \theta_k) \to \left(\bar{\chi}, \bar{\rho}, \bar{\theta}\right)$ such that $(\chi_k, \rho_k, \theta_k) \in \Phi^{-1}(0)$ for every k. By continuity, $\Phi\left(\bar{\chi}, \bar{\rho}, \bar{\theta}\right) = 0$. Hence we are left to check that all $\bar{x}_i$ are interior. Since utility functions are differentiably strictly increasing, the left-hand side of the first order condition $\rho D_{x_i} u_i (x_i; T_i) + (1-\rho) D_{x_i} \hat{u}(x_i) + \varepsilon \alpha_i = \lambda_i p$ is strictly positive and $\bar{p} \gg 0$. We show $\bar{x}_i \notin \Re_+^N \backslash \Re_{++}^N$ for all i in the following. Suppose there is $\bar{x}_{\bar{\imath}\bar{n}} = 0$ for some $\bar{\imath}$ and some $\bar{n}$. Let

$$\begin{aligned} v_i (x_i, \rho, \theta) &= \rho u_i (x_i; T_i) + (1-\rho) \hat{u}(x_i) + \varepsilon \alpha_i x_i, \text{ and} \\ D_{x_{in}} v_i (x_i, \rho, \theta) &= \rho D_{x_{in}} u_i (x_i; T_i) + (1-\rho) D_{x_{in}} \hat{u}(x_i) + \varepsilon \alpha_{in}. \end{aligned}$$

The first order condition says, for all $n \neq \bar{n}$, $D_{x_{\bar{\imath}\bar{n}}} v_{\bar{\imath}} \left(x_{\bar{\imath}}, \bar{\rho}, \bar{\theta}\right) = D_{x_{\bar{\imath}n}} v_{\bar{\imath}} \left(x_{\bar{\imath}}, \bar{\rho}, \bar{\theta}\right) \bar{p}_{\bar{n}} / \bar{p}_n$. By continuity, we can find $n' \in \{1, \ldots, N\} \backslash \bar{n}$, a small ϵ, and two points $\chi' = \left((x_i')_{i=1}^I, \left(\bar{\lambda}_i\right)_{i=1}^I, (y_j)_{j=1}^J, (\bar{\nu}_j)_{j=1}^J, \bar{p} \right)$ and $\chi'' = \left((x_i'')_{i=1}^I, \left(\bar{\lambda}_i\right)_{i=1}^I, (y_j)_{j=1}^J, (\bar{\nu}_j)_{j=1}^J, \bar{p} \right)$ in the neighborhood of χ, where for all $i \neq \bar{\imath}$, $x_i' = x_i'' = \bar{x}_i$, and for $\bar{\imath}$ we have $x_{\bar{\imath}\bar{n}}' = \epsilon$, $x_{\bar{\imath}\bar{n}}'' = \bar{x}_{\bar{\imath}\bar{n}} = 0$, $x_{\bar{\imath}n'}'' > \bar{x}_{\bar{\imath}n'}$, $x_{\bar{\imath}n}' = x_{\bar{\imath}n}'' = \bar{x}_{\bar{\imath}n}$ for all $n \neq \bar{n}, n'$, such that $v_i \left(x_i'', \bar{\rho}, \bar{\theta}\right) \geq v_i \left(x_i', \bar{\rho}, \bar{\theta}\right)$. This violates the boundary condition of utility function since $x_{in}'' \in \Re_+^N \backslash \Re_{++}^N$. ∎

In the following, we show that 0 is a regular value for all these maps Φ, ϕ and η.[3]

Lemma 2. *0 is a regular value for $\Phi(.,.,\theta)$ except for θ in a closed set of measure zero in Θ.*

Proof. We need $D_{(\chi,\rho,\theta)}\Phi$ to have full rank whenever $\Phi(\chi,\rho,\theta)=0$. And

$$D_\theta \Phi = \left[\begin{array}{cccccccccc}
\varepsilon I_N & 0 & & & \cdots & & & & & 0 \\
0 & \ddots & 0 & \cdots & & & & & & \\
 & 0 & -\rho p y_1 I_{N-1} & 0 & & & & & & \\
 & \cdots & 0 & -(\varepsilon\rho+1-\rho)\nu_1 I_N & 0 & \cdots & & & & \\
 & & \cdots & 0 & \varepsilon\nu_j I_N & 0 & \cdots & & & \\
\vdots & & & & & \ddots & & & & \vdots \\
 & \cdots & 0 & (\rho\varepsilon+1-\rho) y_1 & 0 & 0 & \varepsilon\rho+1-\rho & 0 & \cdots & \\
 & & \cdots & 0 & \varepsilon y_j & 0 & 0 & \varepsilon & 0 & \\
 & & & \cdots & 0 & \ddots & & 0 & \ddots & 0 \\
0 & & & & \cdots & & & & 0 & -I_N
\end{array}\right]$$

$$\alpha_i \qquad s_{-1} \qquad \beta_1 \qquad \beta_{j,j\neq 1} \qquad \gamma_1 \qquad \gamma_{j,j\neq 1} \qquad e_1$$

always has full rank. Therefore, $D_{(\chi,\rho,\theta)}\Phi$ always has full rank.

By the transversality theorem (see Guillemin and Pollack 1974, p. 68, and Mas-Colell 1985, p. 320).

Transversality Theorem. *Suppose that $\phi : X \times S \to \Re^m$ is a C^r map where X, S are C^r boundariless manifolds with $r > \max\{0, \dim(X) - m\}$; let $\phi_s(x) = \phi(x,s)$, $\phi_s : X \to \Re^m$. If $y \in \Re^m$ is a regular value for ϕ, then except for s in a set of measure zero in S, y is a regular value for ϕ_s.*

0 is a regular value for $\Phi(.,.,\theta)$ except for θ in a set of measure zero. The set of critical θ such that 0 is not a regular value is closed. Suppose there is a sequence of $\theta_k \in \Theta$ with associated solutions $(\chi_k, \rho_k, \theta_k) \in \Phi^{-1}(0)$ such that $\theta_k \to \bar{\theta}$ and $D_{(\chi,\rho)}\Phi(\chi_k,\rho_k,\theta_k)$ does not have full rank for all k. By Lemma 1, there is a limit point $(\bar{\chi}, \bar{\rho}, \bar{\theta}) \in \Xi \times [0,1]$ such that $(\chi_k,\rho_k,\theta_k) \to (\chi, \bar{\rho}, \bar{\theta})$. By continuity, $\Phi(\chi,\bar{\rho},\bar{\theta}) = 0$ and $D_{(\chi,\rho)}\Phi(\chi,\bar{\rho},\bar{\theta})$ does not have full rank. ∎

[3] For a C^r map $f : M \to N$ between manifolds, $y \in N$ is a *regular value* if $Df(x)$ has full rank for all $x \in f^{-1}(y)$.

Since the above result holds for all $\rho \in [0,1]$, we have $D_{(\chi,\theta)}\phi$ having full rank whenever $\phi(\chi,\theta) = 0$, and $D_{(\chi,\theta)}\eta$ having full rank whenever $\eta(\chi,\theta) = 0$. Thus 0 is a regular value for both ϕ and η. The following Corollary is immediate.

Corollary 1. *0 is a regular value for $\phi(.,\theta)$ and $\eta(.,\theta)$ except for θ in a closed set of measure zero in Θ.*

Next we show that solutions to $\Phi(.,.,\theta) = 0$ can be bounded by a manifold and there is no sequence of solutions that approaches its boundary. Let $B^N(r) = \{x \in \Re^N \mid x \leq r\}$ denote the N-dimensional ball with radius r. $\Re^{IN}_{++} \times \Re^I \times \Re^{JN} \times \Re^J \times S^N$

Lemma 3. *For each $\theta \in \Theta$ there is a manifold*

$$\Xi(\theta) = \left(B^{IN}(\bar{r}_\theta) \cap \Re^{IN}_{++}\right) \times B^{I+J(N+1)}(\bar{r}_\theta) \times S^N \subset \Xi \times [0,1]$$

such that the following holds true:

(i) If $\Phi(\chi,\rho,\theta) = 0$, then $(\chi,\rho) \in \Xi(\theta)$.

(ii) If there is a sequence $(\chi,\rho_k) \to (\chi,\bar{\rho})$ with $\Phi(\chi,\rho_k,\theta) = 0$, then $\chi \notin \partial\Xi(\theta) = cl\Xi(\theta) \backslash \Xi(\theta)$.

Proof. (i) The following defines the maximum amount of the n-good that can be produced by firms in an economy (ρ,θ).

$$\tilde{y}_n(\rho,\theta) = \max_{y_j \in \Re^N} \sum_{j=1}^{J} y_{jn}$$

$$\begin{aligned} &\rho f_1(y_1,T_1,p) + (\rho\varepsilon + 1 - \rho)(\beta_1 y_1 + \gamma_1) = 0, \\ s.t.\ \ &\rho f_j(y_j,T_j,p) + (1-\rho)\hat{f}_j(y_j) + \varepsilon\left(\beta_j y_j + \gamma_j\right) = 0, \forall j \neq 1, \\ &\textstyle\sum_{i=1}^{I} e_{in'} + \sum_{j=1}^{J} y_{jn'} \geq 0, \forall n' \neq n. \end{aligned}$$

It has a unique solution by strict quasiconcavity. Next, let

$$\tilde{x}(\theta) = \max_{n \in \{1,\dots N\}, \rho \in [0.1]} \left[\tilde{y}_n(\rho,\theta) + \sum_{i=1}^{I} e_{in}\right] + 1.$$

This is more than the maximum amount of the n-good potentially available in economy θ for all ρ. Thus, each x_i is bounded by $B^N(\tilde{x}(\theta)) \cap \Re^N_{++}$, and y_j is bounded by $B^N(\tilde{x}(\theta))$.

Since the values of all x_i and y_j are bounded, the multipliers λ_i and ν_j are bounded by the first order conditions in $\Phi(\chi,\rho,\theta) = 0$. Denote their bounds by $\tilde{\lambda}_i(\rho,\theta)$ and

$\tilde{\nu}_j(\rho,\theta)$. Take

$$\bar{r}_\theta = \max\left[\tilde{x}(\theta), \max_{i=1,\ldots,I, j=1,\ldots,J,\rho\in[0.1]}\left[\tilde{\lambda}_i(\rho,\theta), \tilde{\nu}_j(\rho,\theta)\right]\right].$$

We have the manifold $\Xi(\theta)$.

(ii) At the limit $(\chi, \bar{\rho})$, we have $\Phi(\chi, \bar{\rho}, \theta) = 0$. The boundary problem $\chi \in \partial\Xi(\theta)$ only happens when there is zero consumption in $\bar{x}_i$ or a zero price in $\bar{p}$. These are ruled out by Lemma 1. ∎

In the following, we can safely restrict the domain of $\Phi(.,.,\theta)$ to the manifold $\Xi(\theta)$, and show that there is a solution to $\phi(.,\theta) = 0$ for almost all θ.

Lemma 4. *If 0 is a regular value for $\Phi(.,.,\theta)$, $\phi(.,\theta)$ and $\eta(.,\theta)$ at $\theta \in \Theta$, then $\phi(.,\theta) = 0$ has a solution.*

Proof. We apply the following version of the preimage theorem (Guillemin and Pollack 1974, p. 60, also Mas-Colell 1985, p. 38).

Theorem *Let ϕ be a smooth map of a manifold X with boundary onto a boundariless manifold Y, and suppose that both $\phi : X \to Y$ and $\partial\phi : \partial X \to Y$ are transversal with respect to a boundariless submanifold Z in Y. Then the preimage $\phi^{-1}(Z)$ is a manifold with boundary $\partial\{\phi^{-1}(Z)\} = \phi^{-1}(Z) \cap \partial X$, and the codimension of $\phi^{-1}(Z)$ in X equals the codimension of Z in Y.*

We apply this theorem to $\Phi(.,.,\theta)$ with $\Xi(\theta) \times [0,1]$ as X, $\Xi(\theta) \times \{0\} \cup \Xi(\theta) \times \{1\}$ as ∂X, $\Re^{IN+I+J(N+1)+N-1}$ as Y, and the combination of $\Phi(.,0,\theta)$ and $\Phi(.,1,\theta)$ as $\partial\Phi(.,.,\theta)$. Note that map ϕ is transversal to a point z means that z is a regular value for ϕ. Therefore, we have $\Phi(.,.,\theta)$ and $\partial\Phi(.,.,\theta)$ both transversal to 0.

So, $\Phi^{-1}(0,\theta)$ is a 1-dimensional C^1 manifold with boundary, whose boundary is on the boundary of the domain $\Xi(\theta) \times \{0\} \cup \Xi(\theta) \times \{1\}$. We know that there is already a unique boundary point $(\chi', 0) \in \Xi(\theta) \times \{0\}$ where $\eta(\chi', \theta) = 0$. By the classification theorem of 1-dimesional manifolds (Hirsch 1976, p.32 and Guillemin and Pollack 1974, p.64), this boundary point of $\eta(.,\theta) = 0$ is either part of a closed curve diffeomorphic to $[0,1]$, or a half-open curve diffeomorphic to $[0,1)$. Suppose it is a half-open curve. Then, its open end cannot approach the boundary $\partial\Xi(\theta)$ by Lemma 4 (ii), and this open end cannot be in $\Xi(\theta)$ since this violates continuity of Φ. Thus, $\Phi^{-1}(0,\theta)$ is a closed C^1 curve with another end point $(\chi^*, 1) \in \Xi(\theta) \times \{1\}$ where $\phi(\chi^*, \theta) = 0$.[4] ∎

[4] This result can also be obtained from the mod 2 intersection (or degree) theorem, but it requires a

Therefore, generic in θ, there is a solution to $\phi = 0$. Moreover, all critical θ values in Lemma 4, such that 0 is not a regular value, are in a nowhere dense set of Θ (Corollary 1). For a critical $\bar{\theta} \in \Theta$, we can find a sequence $\theta_k \to \bar{\theta}$ such that 0 is a regular value for those maps in Lemma 4 at each θ_k, and each θ_k has an associated equilibrium χ_k. Since Lemma 1 shows that $\Phi^{-1}(0)$ is closed, $(\chi_k, 1, \theta_k)$ converges to an $\bar{\chi}$, and by continuity $\phi\left(\bar{\chi}, \bar{\theta}\right) = 0$ and $\bar{\chi}$ is an equilibrium for $\bar{\theta}$. ∎

3 Conclusion

Our paper presents a differentiable approach to externalities, where convexity in the aggregate is not required. Externalities are allowed to influence consumers and firms in arbitrary ways. Utility and production functions can be nonconvex in externalities, and externalities can be price dependent. As long as preferences and production are convex in own activities for fixed levels of externalities, existence of competitive equilibrium obtains. Our approach is in line with Mas-Colell (1985), Balasko (1988), and Geanakoplos and Shafer (1990), which study equilibria of an economy as the intersection of manifolds. A nonempty intersection obtains if these manifolds are transversal. The fixed point mapping needs not to be convex valued. As long as utility and production functions are convex in own activities, standard assumptions are sufficient for equilibrium.

References

[1] Allen, B. (1981) Utility perturbations and the equilibrium price set. Journal of Mathematical Economics 8, 277–307.

[2] Balasko, Y., 1988. Foundations of the theory of general equilibrium. In: Economic Theory, Econometrics, and Mathematical Economics. Academic Press, Boston, MA

[3] Balder, E. (2004), Existence of Competitive Equilibria in Economies with a Measure Space of Consumers and Consumption Externalities, Mimeo, University of Utrecht.

[4] Baumol, W. (1972), On taxation and the control of externalities, *American Economic Review* 62, 307-322.

more involved operation that extends domain $\hat{\mathcal{D}}(A)$ into a compact manifold without boundary.

[5] Bonnisseau, J.-M. and E. L. del Mercato (2010), Externalities, consumption constraints and regular economies," *Economic Theory* 44(1), 123-147.

[6] Boyd, J. H. and Conley, J. P. (1997), Fundamental Nonconvexities in Arrovian Markets and a Coasian Solution to the Problem of Externalities, *Journal of Economic Theory* 72, 388-407.

[7] Coase, R. (1960), The Problem of Social Cost, *Journal of Law and Economics* 3, 1-44.

[8] Cornet, B. and M. Topuzu (2005), Existence of Equilibria for economies with Externalities and a Measure Space of Consumers, *Economic Theory* 26, 397-421.

[9] Debreu, G. (1952), A Social Equilibrium Existence Theorem, *Proceedings of the National Academy of Sciences of the United States of America* 38, 886-893.

[10] ————- (1970), Economies with a Finite Set of Equilibria, *Econometrica* 38, 387-392.

[11] del Mercato, E. L. (2006), Existence of Competitive Equilibria with Externalities: A Differentiable Viewpoint, *Journal of Mathematical Economics* 42, 525-543.

[12] Dierker, E. (1974), Topological Methods in Walrasian Economics, in Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York, NY.

[13] Dierker, H. (1975), Smooth Preferences and the Regularity of Equilibria, *Journal of Mathematical Economics* 2, 43-62.

[14] Geanakoplos, J. and W. Shafer (1990), Solving Systems of Simultaneous Equations in Economics, *Journal of Mathematical Economics* 19, 69-93.

[15] Guillemin, V. and A. Pollack (1974), Differential Topology, Prentice-Hall, Englewood Cliffs, NJ.

[16] Hammond, P. J., M. Kaneko and M. H. Wooders (1989), Continuum Economies with Finite Coalitions: Core, Equilibria, and Widespread Externalities, *Journal of Economic Theory* 49, 113-134.

[17] Kung, F-C. (2008), Voluntary contributions to multiple public goods in a production economy with widespread externalities, *Journal of Mathematical Economics* 44, 1364–1378.

[18] Ledyard, J. (1971), The Relation of Optima and Market Equilibria with Externalities, *Journal of Economic Theory* 3, 54-65.

[19] Malinvaud, E. (1985), Lectures on Microeconomic Theory, 2^{nd} ed., North-Holland, New York, NY.

[20] Mas-Colell, A. (1985), The Theory of General Economic Equilibrium: A Differential Approach, Cambridge University Press, Cambridge, MA.

[21] Mas-Colell, A. and J. H. Nachbar (1991), On the Finiteness of the Number of Critical Equilibria, with an Application to Random Selections, *Journal of Mathematical Economics* 20, 397-409.

[22] Milleron, J-C. (1972), Theory of Value with Public Goods: A Survey Article, *Journal of Economic Theory* 5, 419-477.

[23] Noguchi, M. and W. R. Zame (2006), Competitive Markets with Externalities, *Theoretical Economics* 1, 143-166.

[24] Shafer, A. and H. Sonnenschein (1975), Equilibrium in Abstract Economies without Ordered Preferences, *Journal of Mathematical Economics* 2, 345-348.

[25] Smale, S. (1974), Global analysis and economics IV: finiteness and stability of equilibria with general consumption sets and production, *Journal of Mathematical Economics* 1, 119-127.

[26] Starrett, D. A. (1972), Fundamental Nonconvexities in the Theory of Externalities, *Journal of Economic Theory* 4, 180-199.