

Competitive General Equilibrium in an Economy where Indivisible Goods Generate Network Externalities*

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Abstract

In the present paper we provide sufficient conditions for existence of competitive equilibrium with NE *and* indivisibilities combined. The key conditions are a large number of consumers and dispersion in their income distribution. A Network Externality arises when the satisfaction that a consumer obtains from the consumption of a given good depends (usually positively) on the number of consumers that consume the same good. A common feature of markets where NE are present is the phenomenon called “tipping”, which is the tendency of one of the competing goods or protocols to win a substantial share of the market. We argue that for tipping to occur there must be an underlying indivisibility in the consumption space. In addition to the problems for existence of equilibrium posed by externalities, indivisibilities create discontinuous demand behavior (not merely a nonconvexity).

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1 Introduction

A Network Externality (NE in what follows) arises when the satisfaction that a consumer obtains from the consumption of a given good depends (usually positively) on the number of consumers that consume the same good.¹ Usual examples are telephone and e-mail services. The fact that a new individual uses the service enhances the usefulness of the service for existing users, since then they can interact with an additional subscriber.

In virtually all the previous NE literature the problem is addressed from a partial equilibrium point of view. Examples of this work are Dybvig and Spatt (1983), Farrell and Saloner (1985, 1986 a, 1986 b and 1988), Katz and Shapiro (1985, 1986 and 1994) and Rohlfs (1974). Discussions about NE are found in Besen and Farrell (1994) and Liebowitz and Margolis (1994). Quite surprisingly, there has been almost no theoretical work on NE in a General Equilibrium framework. The present research fills the gap in the literature by presenting a general equilibrium model of an economy with NE, and showing that a competitive equilibrium exists in that framework.

The only previous work on NE in a General Equilibrium model is Starr (1999), who presents a model in which network goods are produced by firms using a technology characterized by set-up costs. He then shows that an Average Cost Pricing Equilibrium exist in this economy. In the present paper the problem is treated from a different perspective. Whereas in Starr's framework NE are due to cost sharing enjoyed by joining the same network, here we concentrate in the case in which there is a *direct* effect from an individual's consumption of the network goods on people in the same network. The word direct means that this effect does not come through a reduction in price from cost sharing in an Average Cost Pricing setting, but rather that the consumption of network goods by an individual enters other individuals' utility functions.

The problem of externalities has been previously studied in the General Equilibrium literature. Arrow and Hahn (1971) show that in general a competitive equilibrium exists

¹The word "number" is used here in a broad sense here. A consumer may care differently about the addition of two different individuals to the network. However, the assumption that only the number of consumers matter simplifies the analysis greatly and therefore it is adopted often in the literature.

when externalities are present. In their analysis the utility functions and production sets are affected by the allocation of the economy, i.e. by consumption vectors and production vectors of all agents. In equilibrium, each household maximizes their utility subject to what all other households and firms do, and each firm maximizes its profits subject to what all other firms and households do. The equilibrium allocation is therefore consistent in the sense that at that allocation everybody is maximizing (utility or profits) conditional on everybody else's actions. However, as we argue below, the most interesting cases of NE pose additional problems for existence of equilibrium, beyond those addressed by Arrow and Hahn (1971). The goal of the present paper is to clarify those issues and to provide a proof of existence of equilibrium in this NE framework.

A common feature of markets with NE is the phenomenon called “tipping”, which is the tendency for one of the competing goods or protocols to pull away from its competitors and win a substantial share of the market. Perhaps not surprisingly, in virtually all papers where tipping results are found it is implicitly or explicitly assumed that there is an indivisibility intrinsic to the good in question. In these papers typically a set of consumers face the decision of buying one among several indivisible competing goods exhibiting the NE property. The most usual situation is that in which the goods are substitutes, and different network goods are incompatible with each other. Then under that setting consumers try to buy popular goods, giving rise to tipping.

Although in the present article we do not formally address the problem of whether indivisibilities are necessary for tipping behavior to arise,² we intuitively argue that the occurrence of tipping is due not only to NE, which make it advantageous for consumers to buy popular products if they want to interact with other consumers, but also to some indivisibility, which encourages consumers to coordinate on a given network good. In fact, if no indivisibility were present, there would be no need for a consumer to chose only one of the competing goods or protocols. In that case it would be possible for her to buy a mix of many of network goods in the desired proportions, and that would be better for her if variety is something she likes. However, this is not possible for many important situations in which NE are present.

²For some recent treatment on this problem see Heal and Kunreuther (2007).

For instance, it is impossible for a consumer to buy 10 hours of usage of an operating system per month.

The presence of indivisibilities in consumption pose a difficulty for existence of competitive equilibrium, since then the households' consumption sets are non-convex, which implies possibly discontinuous demand behavior, i.e. the households' demand correspondences are not necessarily upper-hemicontinuous. The problem of indivisibilities has been previously addressed in the general equilibrium literature. Yamazaki (1978) shows that in large economies a competitive equilibrium exists even when the individual consumption sets are allowed to be non-convex, as it is the case with indivisibilities in consumption.³ The usual problem for existence of equilibrium in this setting is that individual demand behavior may be discontinuous, since at certain prices a small change in the price vector may cause a large “jump” in the consumption bundle of an agent. The key condition used in Yamazaki (1978) to solve this problem is an assumption on “dispersion of income”. That condition says basically that for any price vector, there is no income level such that a large portion of the population (a set of positive measure) has exactly that income. In Yamazaki's work that assumption allows for the “smoothing” effect of large numbers, which ensures that aggregate behavior is continuous even when individual behavior is not. The intuition goes as follows: In large economies, each consumer is small compared to the size of the population. Then, even when small changes in prices may bring large changes in consumption for a set of agents, under suitable conditions that set of agents will be negligible compared with the size of the population, and therefore the jump in individual consumption will be small compared with aggregate consumption.

It is worth to mention that our problem in the present paper, as well as in Yamazaki (1978), are different from that in Aumann (1966). In Aumann's treatment preferences are allowed to be non-convex, but the consumers' opportunity sets are assumed to be \mathbf{R}_+ (which is of course convex). In that setting individual demand is not necessarily convex valued, but it is upper-hemicontinuous. In contrast, in our setting the consumption sets are assumed to be non-convex, generating possibly discontinuous demands. What Aumann (1966) shows is

³For another treatment of the same problem, see Mas-Colell (1977).

that in markets with a continuum of consumers even though the individual demand correspondence may not be convex valued, the aggregate demand correspondence will be.

In the present paper we provide sufficient conditions for existence of competitive equilibrium with NE *and* indivisibilities combined. The key conditions to ensure existence are a large number of consumers and dispersion in their income distribution, a condition similar to that used in Yamazaki (1978). It is not obvious, though, that results similar to those in Yamazaki's paper will be true in our case. As it was mentioned above, the results in Yamazaki (1978) rely upon the fact that individual behavior is negligible in comparison to aggregate behavior, and that the set of consumers "jumping" to a very different consumption bundle is small for any price vector and a small change in prices. In the case of economies with NE it is not clear that this will be the case. In the presence of NE, when making decisions consumers are not only looking at prices but also at what other consumers do. As mentioned above the occurrence of tipping is a well documented fact in these markets. Now the idea of tipping is opposed to the concept of small numbers, since tipping implies a substantial portion of the population buying the same network good. It is possible then that under these conditions we observe jumps for large groups of consumers, and that could pose a problem for existence of competitive equilibrium.

Another interesting question is the efficiency properties of a competitive equilibrium with NE. This question is not formally answered in the current article. However, our intuition tells us that equilibrium in this case will not in general be Pareto efficient, for two different reasons. First, it is possible that competition between network goods does not end with the market adopting the best protocol as a standard (see for example David (1985)). There are many reasons why an inferior protocol may become the standard, for example protocols may be owned or sponsored by firms that will make an effort for imposing them in the market. Usual examples in the Industrial Organization literature are the competition in the home video market between VHS and Betamax and the competition between the Dvorak and QWERTY keyboards. When the market is dominated by an inferior protocol equilibrium is clearly not Pareto efficient, since it would be advantageous for everyone in the network to switch to the superior protocol (assuming of course that everybody in the network regards

that protocol as superior). But each individual has no incentive to switch unilaterally since that way he would lose the interaction benefits.

Second, it is well known that in the presence of positive (negative) externalities individuals will consume less (more) of the externality than the optimal quantity. This of course is also true in the case of NE, in which consumers do not consider the effect that their consumption has in the value of the network for other consumers. These arguments suggest that if we can show that an equilibrium exists with NE, we can't expect it to be Pareto efficient.

The description of NE above suggests a very close relation with what has been called "Clubs" in the literature. The theory of clubs, started by Buchanan's (1965) seminal article, addresses the question of the efficient provision of certain "imperfect" public goods. The word imperfect comes from the fact that in general exclusion is possible for these goods, and also from the fact that consumption is partially rival, in general due to congestion.⁴ In our case we can think of a NE as a case of "negative congestion", i.e. one in which existing consumers benefit from the addition of new consumers to the club. The analogy is not complete though, since we can say that a new member of the network will enlarge the size of the public good, i.e. that the quantity of the public good is not being held fixed.

In recent years, there has been some interest in exploring existence of equilibrium in club economies. Some of the papers on this topic are Cole and Prescott (1997), Ellickson, Grodal, Scotchmer and Zame (1999) and Scotchmer (1997). A common feature of these papers is that they consider clubs as predetermined consumption units that are small compared with the size of the population. In the case of NE, the assumption of small clubs is not very appealing, since in view of the observations in the literature about tipping, we would expect that a significant portion of the population joins a given "club". Then the existence theorems for club economies with small clubs do not seem applicable to NE. Instead, it would make more sense to look for an existence theorem in which the size of the clubs can be large (and endogenous) compared with the population. For these reasons we can interpret the current paper as an extension to the results on club economies to the case in which club sizes are endogenous and can be of the same order of magnitude as the size of the population.

⁴Other references for this literature are Berglas (1976) and Ng (1973 and 1974). A comprehensive survey is presented in Sandler and Tschirhart (1980).

The balance of the paper is organized as follows. In section 2 we present the model and the main result of the paper, the existence theorem. In section 3 a preliminary lemma is stated, which will be helpful in the proof of the existence theorem. In section 4 we provide the proof. Section 5 contains some remarks and concludes.

2 The model and main result

As we argued above, when we have NE we typically have to deal with both indivisibilities *and* externalities, and in this case it is not obvious that previous results on existence of equilibrium will obtain, due to the tipping behavior that is expected in this situation. In this section we present a model that incorporates both features, indivisibilities and externalities. The model is in the spirit of Yamazaki's, from which some basic results will be taken. However, in our case the problem is more complex and therefore the strategy of proof is not exactly the same.

2.1 Private goods

There are N private goods, which are perfectly divisible. There are no externalities in these goods. They are not produced, they come only from endowment.

2.2 Network goods

There are M different activities that consumers can perform in networks. Access to the networks can be obtained by buying a certain network good, i.e. there are M network goods. Each household's satisfaction from joining a given network depends on the number of households in the network. Then all households are identical from the point of view of others regarding their effect on welfare of households in the networks they join.

There is a measure space of consumers⁵ (A, \mathcal{A}, μ) , where $A = [0, 1]$, \mathcal{A} is the Borel σ -algebra on $[0, 1]$ and μ is Lebesgue measure on measurable subsets of $[0, 1]$. We define a network as a set $B_j \subset [0, 1]$, where B_j is the (measurable) set of consumers joining network

⁵In the present paper we use the terms "consumers" and "households" interchangeably.

$j \in \{1, 2, \dots, M\}$. The assumption above about consumers only concerned with the number of other consumers in each network translates formally into $\mu(B_j)$ entering the individual utility functions. We also assume that only one network will form for each activity. Once a consumer buys a network good, she can interact with everybody who bought the same network good. Network goods are indivisible. For each network good, each household has the choice of either buy or not.

Given the description of private and network goods above, the consumption set for a consumer is

$$X = \mathbf{R}_+^N \times \{0, 1\}^M$$

where the first N components of a consumption vector represent the quantities of private goods consumed and the last M components represent the quantities of network goods.

2.3 Consumers

As stated above, there is a measure space of consumers (A, \mathcal{A}, μ) . Each consumer is endowed with $e(a) \in \mathbf{R}_+^{N+M}$, although only quantities of the N private goods, the first N components in $e(a)$, can be positive in the endowment vector. We assume that mean endowment, $\int_A e(a)$, is finite.

We define a *consumption allocation* and a *total consumption allocation* as:

Definition 1 *A consumption allocation c is an integrable function $c : A \rightarrow \mathbf{R}_+^{N+M}$ such that a.e. in A , $c(a) \in X$. For a consumption allocation c , a total consumption allocation λ is defined as $\lambda = (\int_A c_1, \dots, \int_A c_{N+M})$ (c_1 is the first component of the function c , and so on).*

In the previous definition $c(a)$ is a (vector valued) function specifying in its k^{th} coordinate the consumption of good k for consumer $a \in A$, whereas λ is the vector that results from integrating the consumption allocation c over the set of consumers A . The last M coordinates of the vector λ represent the proportion of the population participating in each of the M networks, since the total measure of the population is equal to 1.

Each consumer's satisfaction depends on her own consumption of private and network goods, and also on the measure of the set of consumers joining the networks she joins. Consumer a 's preferences are represented by the continuous utility function:

$$u(a, x, \lambda) \quad x \in X$$

In what follows, we will assume strict monotonicity in own consumption of all goods, i.e.

$$u(a, x, \lambda) > u(a, x', \lambda) \quad \text{if} \quad x \geq x', x \neq x'.$$

The *endowment distribution* μ_e is defined as the image measure of μ with respect to the mapping e , i.e. for every Borel set $B \in \mathbf{R}^{N+M}$

$$\mu_e(B) = \mu(\{a \in A : e(a) \in B\})$$

The following definition is taken from Yamazaki (1978):

Definition 2 *Denote the price vector $p = (p_N, p_M)$, where p_N represents the first N components of p and p_M its last M components, and let $pe(a)$ be the dot product of the vectors p and $e(a)$. The endowment distribution is said to be dispersed if the resulting wealth distribution from endowment, the measure*

$$\mu_{e,p}(D) = \mu(\{a \in A : pe(a) \in D\})$$

on $\mathcal{B}(\mathbf{R})$ is absolutely continuous with respect to Lebesgue measure on \mathbf{R} for each $p \in \mathbf{R}_+^{N+M}$, $p_N \neq 0$.

In Yamazaki (1978) the endowment distribution is assumed to be dispersed. Intuitively, this dispersion assumption says that the measure $\mu_{e,p}$ does not give positive measure to any particular wealth value $w \in \mathbf{R}$. In Yamazaki's work, which considers an exchange economy, this assumption is helpful in showing existence of equilibrium. The reason is because it prevents large sets of consumers from having the same income (for some prices) and therefore behaving similarly. Since individual behavior may not be continuous when the consumption set is not convex, existence of equilibrium relies heavily on the smoothing of

aggregate behavior due to “large numbers”. In economies with uncountably many consumers we can have continuous aggregate demand behavior even when individual demand is not. But if large sets of consumers exhibit jumps in demand at the same time, we may find non-continuous aggregate demand. In Yamazaki (1978) dispersion of endowments ensures that this is not the case.

In our model, however, income does not come only from endowment, since there are firms that produce network goods and are owned by households. Therefore in our case the endowment distributional assumption above will not be sufficient. In section 2.7 we introduce an appropriate condition for our economy, that will ensure that income from all sources is dispersed.

2.4 Firms

Private goods are not produced, they come only from endowment, and are used for consumption and as inputs in the production of network goods. There is a finite set F of firms, with M firms in it, each producing one network good using private goods as inputs. We assume that these firms behave competitively, despite the fact that there is only one firm producing each network good.

Firm $f \in F$ is endowed with a production set $Y^f \in \mathbf{R}^{N+M}$. For a production vector $y \in Y^f$, its first N components represent private goods, which are used as inputs and therefore they are nonpositive, while the last M components represent network goods, which are outputs for the firms, and therefore nonnegative. We define a *production allocation* as:

Definition 3 *A production allocation is a function $s : F \rightarrow \mathbf{R}^{N+M}$ such that for all $f \in F$, $s(f) \in Y^f$.*

The following assumptions on Y^f will be maintained throughout the paper:

1. $0 \in Y^f$
2. Y^f is closed
3. Y^f is convex.

We define the sum of sets $A + B$ as the set such that $x \in A + B$ if there is $y \in A$ and $z \in B$ such that $x = y + z$. The set of *total production allocations* is the set $Y = \sum_F Y^f$, i.e. it is the economy's production possibility set. An assumption on Y will be helpful in proving the main result of the paper (the assumption is only A.1, since A.2 and A.3 are a consequences of A.1 and assumptions 1-3 above):

$$(A.1) \ Y \cap \mathbf{R}_+^{N+M} = \{0\} \text{ (no free lunch)}$$

Since the first N components of $y \in Y^f$ are nonpositive and the last M nonnegative, it follows from (A.1) that

$$(A.2) \ Y \cap -Y = \{0\} \text{ (irreversibility)}$$

This is so since a good that is an input for a given firm cannot be produced by any other firm.

We say that an aggregate production vector $y = \sum_F y^f$ is *attainable* if $y + \int_A e(a) \geq 0$. Standard arguments (see for example Starr 1997, theorem 8.2, pp. 114) ensure that the set of attainable vectors is compact. We summarize that result in the following condition:

$$(A.3) \ \text{The set of attainable vectors in } Y \text{ is compact.}$$

2.5 Prices

The price space, denoted by PS , will be the $N + M$ -dimensional unit simplex.

2.6 Supply behavior

For a given price vector $p \in PS$ and a production vector $y \in Y^f$ profits are given by the inner product py . The objective of firms is to maximize profits subject to production sets, taking prices as given. The supply set of firm f at prices p is

$$S(f, p) = \{y \in Y^f : py \geq pz \text{ for all } z \in Y^f\}.$$

Notice that we assume that there are no externalities in production. The profit function for firm f , that is, the function that specifies the maximum profit for the firm for each price vector $p \in PS$, is denoted by $\Pi(f, p)$.

2.7 Income and demand behavior

Households' income comes from 2 sources, endowment and shares on firms' profits. Households' shares on firm f 's profits are given by the measurable function $\alpha(a, f)$, where for a set of households $B \in \mathcal{A}$, $\int_B \alpha(a, f)$ represents set B 's share on firm f 's profits, $\alpha(a, f) \geq 0$ a.e. in A and $\int_A \alpha(a, f) = 1$. Income for household a at prices p is defined as

$$M(a, p) = pe(a) + \sum_{f \in F} \alpha(a, f) \Pi(f, p)$$

As it was mentioned above, in proving the main result of the paper a distributional assumption on individual incomes will be useful. The following example shows why in our case the assumption on the distribution of endowment used in Yamazaki (1978) is not sufficient to ensure dispersion in the distribution of income:

Example 1 Consider an economy with one private good and one network good. Price of the private good is equal to 1. $A = [0, 1]$ and $e_1(a) = a$ for $a \in A$ (e_1 denotes the first component of the endowment vector e). Profits for the only firm are equal to $\frac{1}{2}$ and shares on firms profits are $\alpha(a, f) = 2 - 2a$ for $a \in A$. Then the endowment distribution is dispersed, but income is equal to 1 for all $a \in A$.

As seen in the example, dispersion of endowments is not sufficient to ensure dispersion of incomes. In the example the problem arises because the distribution of ownership shares is correlated with the distribution of endowment.

The *income distribution* for $p \in PS$ is defined as the image measure of μ with respect to the mapping M , i.e.

$$\mu_{M,p}(B) = \mu(\{a \in A : M(a, p) \in B\})$$

for every Borel set $B \subset \mathbf{R}$. Throughout the paper we will impose the following assumption on the income distribution:

ASSUMPTION 1: We assume that the income distribution is dispersed, this is, the measure $\mu_{M,p}$ is absolutely continuous with respect to Lebesgue measure on \mathbf{R} for each $p \in PS$, $p \neq 0$.

Notice that we may have a large portion of the population (possibly all of it) in a small income interval and still satisfy the dispersion condition. All that is required is that there

is no set of consumers having positive measure with *exactly* the same income for any price vector $p \in PS$. More formally, assumption 1 implies that the distribution of income is non-atomic.

The budget set for consumer $a \in A$ at prices $p \in PS$ is:

$$B(a, p) = \{x \in X : px \leq M(a, p)\}$$

For a given λ , consumers are assumed to maximize $u(a, x, \lambda)$ subject to consuming within their budget set. The demand set for agent a at prices p and total consumption allocation λ is:

$$D(a, p, \lambda) = \{x \in B(a, p) : u(a, x, \lambda) \geq u(a, z, \lambda) \text{ for all } z \in B(a, p)\}$$

2.8 Equilibrium

An *allocation* is a pair (c, s) where c is a consumption allocation and s is a production allocation. Next, we define competitive equilibrium in our economy:

Definition 4 *A competitive equilibrium is defined as a price vector $p^* \in PS$, and an allocation (c^*, s^*) and a total consumption allocation ρ such that*

1. $c^*(a) \in D(a, p^*, \rho)$ a.e. in A
2. $s^*(f) \in S(f, p^*)$ for all $f \in F$
3. $\int_A c^* \leq \sum_F s^* + \int_A e$
4. $\rho = \lambda^* (= \int_A c^*)$

The last condition for an equilibrium is important. It says that in equilibrium, agents must be optimizing *given* the equilibrium consumption choices of other agents. The main result of the paper is stated in the following theorem, which is proven in section 4.

Theorem 1 *In the economy specified above there exists a competitive equilibrium.*

3 Preliminary result

In this section we present a result that will be helpful in proving theorem 1. For a proof of this result, the reader is directed to Yamazaki (1978). The proof given there does not rely on any property of the consumption set and therefore applies here as well.

A known problem for existence of equilibrium, referred to in the literature as the “Arrow corner”, arises when the income of some household is equal to zero at some price vector, making their demand behavior possibly discontinuous. The traditional solution is to bound income away from zero. However, when the economy is “large”, in the sense that there are uncountably many consumers, our previous assumption on dispersion of income will in turn ensure that the set of households that could exhibit the Arrow corner for any price vector is at worst a set of measure zero, and therefore does not matter for aggregate demand. The result in this section tells us that this is indeed the case.

We say that a consumption bundle $x \in X$ has *local cheaper points* at prices $p \in PS$ if for any neighborhood $U(x)$ there exist $z \in U(x) \cap X$ such that $pz < px$. For $p \in PS$ we define

$$C^p = \{x \in X : x \text{ does not have local cheaper points at prices } p\}$$

$$A^p = \{a \in A : \text{there is a vector } x \in C^p \text{ such that } px = M(a, p)\}$$

The set A^p is the set of households whose “budget line” contains points in C^p . The following lemma, taken from Yamazaki’s Corollary 1 (1978, pp. 549), asserts that if the income distribution is dispersed we don’t have to worry for the behavior of households in the set A^p .

Lemma 1 *If the distribution of income is dispersed, the set A^p is a set of measure zero for all $p \in PS$.*

4 Proof of the existence theorem

In this section we will present the proof for the main result of the paper, namely that in the model presented in section 2 there exist a competitive equilibrium. Before going to the proof, we will introduce some notation and definitions. We already defined a consumption allocation as a measurable function $c : A \rightarrow \mathbf{R}^{N+M}$. Call C the set of all possible consumption allocations, that is, C is the set of all measurable functions $c : A \rightarrow \mathbf{R}^{N+M}$ such that

$c(a) \in X$ a.e. in A . The set of *total consumption allocations* will be denoted by Ω . A total consumption allocation $\lambda \in \Omega$ if there is a consumption allocation $c \in C$ such that $\lambda = (\int_A c_1, \dots, \int_A c_{N+M})$. Note that in any $\lambda \in \Omega$ the first N coordinates indicate aggregate demand for the N private goods, whereas the last M coordinates indicate the proportion of the population that participates in each of the networks. Therefore

$$\Omega = \mathbf{R}_+^N \times [0, 1]^M$$

Finally, define the set Δ as

$$\Delta = \Omega - Y - \left\{ \int_A e \right\}.$$

The strategy of proof will be the following. First we construct the correspondences

1. Total demand correspondence: $\Lambda : PS \times \Omega \rightarrow \Omega$

$$\Lambda(p, \lambda) = \int_A D(a, p, \lambda)$$

2. Excess demand correspondence: $Z : PS \times \Omega \rightarrow \Delta$

$$Z(p, \lambda) = \int_A D(a, p, \lambda) - \int_A e - \sum_F S(f, p) = \Lambda(p, \lambda) - \int_A e - \sum_F S(f, p)$$

3. Price adjustment correspondence: $\rho : \Delta \rightarrow PS$

$$\rho(z) = \{p^* \in PS : p^* = \arg \max_{p \in PS} pz\}$$

Then the correspondence $\rho \times \Lambda \times Z : PS \times \Omega \times \Delta \rightarrow PS \times \Omega \times \Delta$. We will find conditions under which a fixed point theorem can be applied to this correspondence, and after that we will show that the fixed point is a competitive equilibrium for our economy. In fact, we will not carry over this procedure directly. Instead, we will find an equilibrium for a sequence of suitably bounded economies, indexed by $k = 1, 2, \dots$. We will then find a limit point for the sequence of competitive equilibria of the bounded economies, and we will show that this limit point is an equilibrium for the unbounded economy. In constructing a sequence of bounded economies, we will use the following quantity:

$$b = \int_A e_1 + \int_A e_2 + \dots + \int_A e_N$$

The quantity b is the aggregate endowment of *all* private goods combined, and it will be used in truncating the consumption and production sets later. Notice that if the distribution of income is dispersed, then $b > 0$. This is so since if $b = 0$, this implies that $e_i(a) = 0$ for a.e. $a \in A$, $i \in \{1, 2, \dots, N\}$, and since then no production is possible, profits are zero for all firms too. But this contradicts the distribution of income being dispersed, since income is equal to zero a.e. in A .

4.1 Aggregate supply correspondence

The supply correspondence for firm $f \in F$ is:

$$S(f, p) = \{y \in Y^f : py \geq pz \text{ for all } z \in Y^f\}$$

This correspondence may be empty, since the set Y^f is in general unbounded for $f \in F$. To ensure that $S(f, p)$ is nonempty we truncate the production set for firm f . For $k = 1, 2, \dots$, we define the k^{th} truncated production set for firm f as

$$Y^{f,k} = \{y \in Y^f : -kb \leq y_i \leq 0 \text{ for } i = 1, 2, \dots, N, 0 \leq y_i \leq k \text{ for } i = N + 1, \dots, N + M\}$$

Denoting

$$I^k = [-kb, 0]^N \times [0, k]^M \quad k = 1, 2, \dots$$

the k^{th} truncated production set is $Y^{f,k} = Y^f \cap I^k$. It is easily seen that the set $Y^{f,k}$, for $k = 1, 2, \dots$ and all $f \in F$ is: (i) convex, since Y^f and I^k are convex; (ii) compact, since Y^f is closed and I^k is compact; and (iii) $0 \in Y^{f,k}$. We define the supply correspondence for firm f in the k^{th} truncated economy as

$$S^k(f, p) = \{y \in Y^{f,k} : py \geq pz \text{ for all } z \in Y^{f,k}\},$$

the aggregate supply correspondence in the k^{th} truncated economy as

$$\sum_{f \in F} S^k(f, p)$$

and the profit function for firm f in the k^{th} truncated economy as

$$\Pi^k(f, p) = py \quad s.t. \quad y \in S^k(f, p)$$

Lemma 2 *The aggregate supply correspondence in the k^{th} truncated economy, $\sum_{f \in F} S^k(f, p)$, is nonempty, upper-hemicontinuous and convex valued. Furthermore, the profit function $\Pi^k(f, p)$ is a continuous function of p for all $p \in PS$.*

Proof: First, $S^k(f, p)$ is nonempty for all $f \in F$, since it is the set of maximizers of a continuous function on the compact set $Y^{f,k}$. Therefore $\sum_{f \in F} S^k(f, p)$ is the finite sum of nonempty sets and therefore it is nonempty.

Next we show that $S^k(f, p)$ is convex valued. Suppose that $y_1, y_2 \in S^k(f, p)$. Then $py_1 = py_2$. As $Y^{f,k}$ is convex, $\alpha y_1 + (1 - \alpha)y_2 \in Y^{f,k}$, for $\alpha \in [0, 1]$ and $p[\alpha y_1 + (1 - \alpha)y_2] = \alpha py_1 + (1 - \alpha)py_2 = py_1$. Therefore $\alpha y_1 + (1 - \alpha)y_2 \in S^k(f, p)$. Since $\sum_{f \in F} S^k(f, p)$ is the finite sum of convex sets it is also convex.

Now we show that $S^k(f, p)$ is upper-hemicontinuous. Take a sequence $p^n \rightarrow p$, $y^n \rightarrow y$, $y^n \in S^k(f, p^n)$. We want to show that $y \in S^k(f, p)$. Suppose that this is not the case. Then there exist $y' \in Y^{f,k}$, such that $py' > py$. Notice that

$$p^n y' \rightarrow py'$$

$$p^n y^n \rightarrow py$$

then for some $N \in \mathbf{N}$, $n \geq N$ implies

$$p^n y' > p^n y^n$$

which is a contradiction to the fact that $y^n \in S^k(f, p^n)$ for all n . The contradiction proves upper-hemicontinuity of $S^k(f, p)$. It is a well known result that the finite sum of upper-hemicontinuous correspondences is itself upper-hemicontinuous. Therefore $\sum_{f \in F} S^k(f, p)$ is upper-hemicontinuous.

Finally we show that $\Pi^k(f, p)$ is continuous in p . Take a sequence $p^n \rightarrow p$, $y^n \in S^k(f, p^n)$. Without loss of generality take a convergent subsequence $y^j \rightarrow y$ (remember Y^k is bounded). By upper-hemicontinuity of $S^k(f, p)$ we have that $y \in S^k(f, p)$. Therefore

$$p^j y^j \rightarrow py$$

which is

$$\Pi^k(f, p^j) \rightarrow \Pi^k(f, p)$$

Since this is true for any convergent subsequence y^j continuity is proven. Q.E.D.

We now define household income in the bounded economy and show that it is continuous in prices. Household income in the k^{th} economy is defined as

$$M^k(a, p) = pe(a) + \sum_F \alpha(a, f) \Pi^k(f, p)$$

where $\int_A \alpha(a, f) = 1$ for all $f \in F$. We already showed that $\Pi^k(f, p)$ is continuous. It follows trivially that $M^k(a, p)$ is continuous in p .

4.2 Aggregate demand correspondence

Household demand at prices p and total consumption allocation λ was defined above as the set $D(a, p, \lambda)$. This correspondence may be empty, since for some prices the set $B(a, p)$ may be unbounded. In order to overcome this problem we truncate the consumption set X . The k^{th} truncated consumption set is

$$X^k = \{x \in X : x \leq (kb, \dots, kb, 1, \dots, 1)\}$$

and the truncated budget set is defined as

$$B^k(a, p) = \{x \in X^k : px \leq M^k(a, p)\}$$

We define the subset of households whose endowments are in the truncated consumption set

$$A^k = \{a \in A : e(a) \leq (kb, \dots, kb, 0, \dots, 0)\}$$

We will argue that A^k are of positive measure for $k = 1, 2, \dots$. In fact $A^k \subset A^{k+1}$, $k = 1, 2, \dots$, so we only have to show that A^1 is of positive measure. Suppose not. Then there is $i \in 1, 2, \dots, N$ such that $e_i(a) > b$ a.e. in A . Therefore

$$\int_A e_i > b = \int_A e_1 + \int_A e_2 + \dots + \int_A e_N \geq \int_A e_i$$

which is a contradiction. We define the induced measure spaces of consumers $(A^k, \mathcal{A}^k, \mu^k)$, where \mathcal{A}^k is the Borel σ -algebra of measurable subsets of A^k . The measure μ^k is the restriction of μ to sets in \mathcal{A}^k .

In section 2 we defined the set of consumption allocations C . The set of *truncated consumption allocations* C^k , $k = 1, 2, \dots$, is the subset of C such that $c(a) \in X^k$ a.e. in A^k . The set of *truncated total consumption allocations* Ω^k is the subset of Ω such that $\lambda \in \Omega^k$ if there is $c \in C^k$ such that $\lambda = \int_{A^k} c$. The set Ω^k so defined will then be the interval

$$\Omega^k = [0, \mu(A^k)kb]^N \times [0, \mu(A^k)]^M$$

This set is trivially compact and convex. For $\lambda \in \Omega^k$ and $p \in PS$ the *individual truncated demand correspondence* is

$$D^k(a, p, \lambda) = \{x \in B^k(a, p) : u(a, x, \lambda) \geq u(a, z, \lambda) \text{ for all } z \in B^k(a, p)\}$$

and the *aggregate truncated demand correspondence* is

$$\Lambda^k(p, \lambda) = \int_{A^k} D^k(a, p, \lambda)$$

Lemma 3 *The aggregate truncated demand correspondence Λ^k is: (i) nonempty and convex valued for all $(p, \lambda) \in PS \times \Omega^k$ and (ii) Upper-hemicontinuous in (p, λ) .*

Proof: (i) For $(p, \lambda) \in PS \times \Omega^k$, we show that truncated individual demand correspondence is nonempty a.e. in A^k . $B^k(a, p)$ is nonempty a.e. in A^k . It is also compact, since it is closed and it is a subset of X^k which is bounded. Then the truncated individual demand correspondence $D^k(a, p, \lambda)$ is the set of maximizers of the continuous function $u(a, x, \lambda)$ on a compact set and therefore it is nonempty a.e. in A^k . The argument in Hildenbrand (1970, theorem 3 pp. 614), which applies equally here, ensures that $D^k(\cdot, p, \lambda)$ is measurable for $(p, \lambda) \in PS \times \Omega^k$. Therefore

$$\Lambda^k(p, \lambda) = \int_{A^k} D^k(a, p, \lambda)$$

is nonempty for $(p, \lambda) \in PS \times \Omega^k$. Furthermore, it is well known that the integral of a correspondence with respect to an atomless measure is convex valued (see for example Hildenbrand (1970, pp. 615)). Therefore $\Lambda^k(p, \lambda)$ is convex for $(p, \lambda) \in PS \times \Omega^k$.

(ii) We now show that $\Lambda^k(p, \lambda)$ is upper-hemicontinuous. Analogously to section 2, we define the sets

$C^{p,k} = \{x \in X^k : x \text{ does not have local cheaper points at prices } p\}$

$A^{p,k} = \{a \in A^k : \text{there is a point } x \in C^{p,k} \text{ such that } px = M^k(a, p)\}$

It was stated in section 3 that A^p is a set of measure zero. But $A^{p,k} \subset A^p$, so $A^{p,k}$ is also a set of measure zero.

Take the sequence $(p^n, \lambda^n) \rightarrow (p, \lambda), \gamma^n \rightarrow \gamma, \gamma^n \in \Lambda^k(p^n, \lambda^n)$. We want to show that $\gamma \in \Lambda^k(p, \lambda)$. For each n there exist an integrable function $c^n : A^k \rightarrow \mathbf{R}^{N+M}$ such that $c^n(a) \in D^k(a, p^n, \lambda^n)$ a.e. in A^k and $\int_{A^k} c^n = \gamma^n$. Let $L(a)$ be the set of cluster points of the sequence $\{c^n(a)\}$. Notice that $L(a)$ is nonempty since the sequence $\{c^n(a)\}$ is bounded. We will show that $L(a) \subset D^k(a, p, \lambda)$ a.e. in $A^k/A^{p,k}$. This in turn implies that $\int_{A^k} L(a) \subset \int_{A^k} D^k(a, p, \lambda)$. Then by showing that $\gamma \in \int_{A^k} L(a)$ we'll be done.

To show $L(a) \subset D^k(a, p, \lambda)$ a.e. in $A^k/A^{p,k}$, take $a \in A^k/A^{p,k}$ such that $c^n(a) \in D^k(a, p^n, \lambda^n)$ and take a convergent subsequence $c^n(a) \rightarrow g(a)$ (i.e. $g(a) \in L(a)$). Then $g(a) \leq (kb, \dots, kb, 1, \dots, 1)$ since $c^n(a) \leq (kb, \dots, kb, 1, \dots, 1)$ for all n . Now $p^n c^n(a) \leq M^k(a, p^n)$ and $p^n \rightarrow p$ implies $pg(a) \leq M^k(a, p)$. Now take $y \in X^k$ such that $py < M^k(a, p)$. Then $p^n y < M^k(a, p^n)$ for n sufficiently large, and therefore

$$u(a, c^n(a), \lambda^n) \geq u(a, y, \lambda^n)$$

for n sufficiently large. By continuity of u in both arguments we have

$$u(a, g(a), \lambda) \geq u(a, y, \lambda)$$

Now take $y \in X^k$ such that $py = M^k(a, p)$. Since $a \in A^k/A^{p,k}$, there exist a sequence $y^i \rightarrow y$ such that $py^i < M^k(a, p)$ for all i . Then for each i we have

$$u(a, g(a), \lambda) \geq u(a, y^i, \lambda)$$

and by continuity of u we have

$$u(a, g(a), \lambda) \geq u(a, y, \lambda)$$

Therefore $g(a) \in D^k(a, p, \lambda)$ and

$$\int_{A^k} L(a) \subset \int_{A^k} D^k(a, p, \lambda)$$

as was to be shown. Finally we show that $\gamma \in \int_{A^k} L(a)$. Since $c^n(a)$ are bounded by the constant function $h(a) = (kb, \dots, kb, 1, \dots, 1)$, we can use the result in Aumann (1976) that states:

Result 1 (Aumann, 1976 pp. 16) *Let c^n be a sequence of measurable functions from A^k into \mathbf{R}^{N+M} , which are bounded by the integrable function h . Then each cluster point of $\int_{A^k} c^n$ ($= \gamma^n$) belongs to $\int_{A^k} L$. (i.e. γ , which is a cluster point of the sequence $\int_{A^k} c^n$, is in $\int_{A^k} L$).*

Then we have shown that $\gamma \in \Lambda^k(p, \lambda)$. Consequently we have shown that $\Lambda^k(p, \lambda)$ is upper-hemicontinuous and the proof is complete. Q.E.D.

4.3 Excess demand correspondence

At the beginning of this section we defined the excess demand correspondence as $Z : PS \times \Omega \rightarrow \Delta$

$$Z(p, \lambda) = \int_A D(a, p, \lambda) - \int_A e - \sum_F S(f, p) = \Lambda(p, \lambda) - \int_A e - \sum_F S(f, p)$$

The set Δ was defined as

$$\Delta = \Omega - Y - \left\{ \int_A e \right\}.$$

Similarly, we define the set Δ^k as

$$\Delta^k = \Omega^k - Y^k - \left\{ \int_{A^k} e \right\}$$

The set Δ^k is compact, since Ω^k and Y^k are compact. Δ^k is also convex since Ω^k and Y^k are convex. The *truncated excess demand correspondence* $Z^k : PS \times \Omega^k \rightarrow \Delta^k$ is defined as

$$Z^k(p, \lambda) = \int_{A^k} D^k(a, p, \lambda) - \int_{A^k} e - \sum_F S^k(f, p) = \Lambda^k(p, \lambda) - \int_{A^k} e - \sum_F S^k(f, p)$$

Lemma 4 *The truncated excess demand correspondence $Z^k(p, \lambda)$ is upper-hemicontinuous and convex valued.*

Proof: It has already been shown that $\Lambda^k(p, \lambda)$ and $\sum_{f \in F} S^k(f, p)$ are upper-hemicontinuous and convex valued for all $(p, \lambda) \in PS \times \Omega^k$, and since $\int_{A^k} e$ is a real valued vector, then $Z^k(p, \lambda)$ is a sum of upper-hemicontinuous and convex valued correspondences. Therefore it is upper-hemicontinuous and convex valued. Q.E.D.

4.4 Price adjustment correspondence

At the beginning of this section the price adjustment correspondence $\rho : \Delta \rightarrow PS$ was defined as

$$\rho(z) = \{p^* \in PS : p^* = \arg \max_{p \in PS} pz\}$$

Similarly we define the *truncated price adjustment correspondence* $\rho^k : \Delta^k \rightarrow PS$ as

$$\rho^k(z) = \{p^* \in PS : p^* = \arg \max_{p \in PS} pz\}$$

Lemma 5 *The truncated price adjustment correspondence $\rho^k(z)$ is nonempty, upper-hemicontinuous and convex valued.*

Proof: We have to show that ρ^k is upper-hemicontinuous and convex valued (nonempty is trivial). To show convexity, take $z \in \Delta^k$ such that $p_1, p_2 \in \rho^k(z)$. Then $p_1 z = p_2 z = [\sigma p_1 + (1 - \sigma)p_2]z$, $\sigma \in [0, 1]$. Therefore $\sigma p_1 + (1 - \sigma)p_2 \in \rho^k(z)$.

To show upper-hemicontinuity, take a sequence $z^n \rightarrow z, p^n \rightarrow p$, such that $p^n \in \rho^k(z^n)$. Then we have that for all n

$$p^n z^n \geq p' z^n \text{ for all } p' \in PS$$

Taking limits we get

$$pz \geq p' z \text{ for all } p' \in PS$$

which implies that $p \in \rho^k(z)$. Then $\rho^k(z)$ is upper-hemicontinuous. Q.E.D.

4.5 Existence of equilibrium in the truncated economy

In subsection 2.8 we defined of a competitive equilibrium in the unrestricted economy. Similarly, we define a competitive equilibrium in the k^{th} truncated economy as:

Definition 5 *A competitive equilibrium in the k^{th} truncated economy is a price vector $p^{*k} \in PS$, and an allocation (c^{*k}, s^{*k}) such that*

- (1) $c^{*k}(a) \in D(a, p^{*k}, \lambda^{*k})$ a.e. in A^k

- (2) $s^{*k}(f) \in S(f, p^{*k})$ for all $f \in F$
(3) $\int_{A^k} c^{*k} \leq \sum_F s^{*k} + \int_{A^k} e$
(4) $\int_{A^k} c^{*k} = \lambda^{*k}$ (i.e. λ^{*k} is consistent)

Theorem 2 *There exists a competitive equilibrium in the k^{th} truncated economy, $k = 1, 2, 3, \dots$*

Proof: We have shown that the correspondence $\rho^k \times \Lambda^k \times Z^k : PS \times \Omega^k \times \Delta^k \rightarrow PS \times \Omega^k \times \Delta^k$ is upper-hemicontinuous and convex valued. The set $PS \times \Omega^k \times \Delta^k$ is compact and convex. By Kakutani's Fixed Point Theorem, for $k = 1, 2, \dots$, there exist a fixed point $(p^{*k}, \lambda^{*k}, z^{*k})$ such that $(p^{*k}, \lambda^{*k}, z^{*k}) \in \rho^k(z^{*k}) \times \Lambda^k(p^{*k}, \lambda^{*k}) \times Z^k(p^{*k}, \lambda^{*k})$. Now we will show that $(p^{*k}, \lambda^{*k}, z^{*k})$ is a competitive equilibrium in the k^{th} economy.

Since $z^{*k} \in Z^k(p^{*k}, \lambda^{*k})$ and

$$Z^k(p^{*k}, \lambda^{*k}) = \int_{A^k} D(a, p^{*k}, \lambda^{*k}) - \int_{A^k} e - \sum_F S^k(f, p^{*k})$$

there exist integrable functions $c^{*k} : A^k \rightarrow \Omega^k$ and $s^{*k} : F \rightarrow Y^k$ such that $c^{*k}(a) \in D^k(a, p^{*k}, \lambda^{*k})$ a.e. in A^k and $s^{*k}(f) \in S^k(f, p^{*k})$ for all $f \in F$ and

$$z^{*k} = \int_{A^k} c^{*k}(a) - \int_{A^k} e(a) - \sum_F s^{*k}(f)$$

Since $c^{*k}(a) \in D^k(a, p^{*k}, \lambda^{*k})$ a.e. in A^k

$$p^{*k} c^{*k}(a) \leq M^k(a, p^{*k}) = p^{*k} e(a) + \sum_F \alpha(a, f) p^{*k} s^{*k}(f)$$

Integrating over A^k we have

$$p^{*k} \int_{A^k} c^{*k}(a) \leq p^{*k} \int_{A^k} e(a) + \sum_F p^{*k} s^{*k}(f) \int_{A^k} \alpha(a, f)$$

$$p^{*k} \int_{A^k} c^{*k}(a) \leq p^{*k} \int_{A^k} e(a) + p^{*k} \sum_F s^{*k}(f)$$

since $\int_{A^k} \alpha(a, f) \leq 1$ and $p^{*k} s^{*k} \geq 0$. Therefore

$$p^{*k} \int_{A^k} c^{*k} - p^{*k} \int_{A^k} e(a) - p^{*k} \sum_F s^{*k}(f) \leq 0$$

or equivalently $p^{*k} z^{*k} \leq 0$. Since p^{*k} maximizes pz over PS , this implies that $z^{*k} \leq 0$. This is so because if $z_i^{*k} > 0$ for some $i \in \{1, 2, \dots, N + M\}$, then we would have $p^{*k} z^{*k} > 0$.

Finally, we have to show that total demand is consistent, in the sense that in equilibrium individuals are maximizing *given* the equilibrium choices of other agents. But by $(p^{*k}, \lambda^{*k}, z^{*k})$ being a fixed point of the correspondence $\rho^k \times \Lambda^k \times Z^k$ we automatically have that $\lambda^{*k} \in \Lambda^k(p^{*k}, \lambda^{*k})$, which is the consistency condition of equilibrium. Therefore the price vector p^{*k} and the allocation (c^{*k}, s^{*k}) are a competitive equilibrium in the k^{th} truncated economy for $k = 1, 2, \dots$ Q.E.D.

4.6 Existence of equilibrium in the unbounded economy

In this section we finish the proof of the main theorem by showing that there exist an equilibrium in the unrestricted economy. The strategy of proof will be the following: We will take a sequence of equilibria $p^{*k}, (c^{*k}, s^{*k})$ for $k = 1, 2, \dots$, and find a cluster point for this sequence. Then we will show that this cluster point is an equilibrium in the unrestricted economy.

For $k = 1, 2, \dots$, define the function:

$$\begin{aligned} g^k(a) &= c^{*k}(a) \text{ if } a \in A^k \\ &= e(a) \text{ if } a \notin A^k \end{aligned}$$

Since for $k = 1, 2, \dots$

$$\int_{A^k} c^{*k}(a) - \int_{A^k} e(a) - \sum_F s^{*k}(f) \leq 0$$

then

$$\int_A g^k(a) - \int_A e(a) - \sum_F s^{*k}(f) \leq 0.$$

Now we will argue that the sequences $\{\int_A g^k(a)\}$ and $\{\sum_F s^{*k}(f)\}$ are bounded. Rearranging terms

$$0 \leq \int_A g^k(a) \leq \int_A e(a) + \sum_F s^{*k}(f), \tag{1}$$

and therefore

$$\sum_F s^{*k}(f) + \int_A e(a) \geq 0 \quad (2)$$

Notice that equation (2) implies that each of the elements of the sequence $\{\sum_F s^{*k}(f)\}$ is attainable. Therefore, by (A.3) we conclude that the sequence $\{\sum_F s^{*k}(f)\}$ is bounded. Therefore, by equation (1), we conclude that the sequence $\{\int_A g^k(a)\}$ is also bounded.

The following result, which is a consequence of Fatou's lemma in $M + N$ dimensions, was taken from Yamazaki (1978, pp. 552):

Result 2 *Without loss of generality assume $p^{*k} \rightarrow p$ as $k \rightarrow \infty$. Then there exist functions $g : A \rightarrow \mathbf{R}^{N+M}$ (integrable), and $s : F \rightarrow \mathbf{R}^{N+M}$, such that*

$$\int_A g(a) - \int_A e(a) - \sum_F s(f) \leq 0$$

and $(g(a), s(f))$ is a cluster point of $(g^k(a), s^{*k}(f))$ a.e. in A and for all $f \in F$.

Let $\lambda = \int_A g$. Then $\lambda^{*k} \rightarrow \lambda$ as $k \rightarrow \infty$. The proof of the following lemma completes the argument:

Lemma 6 *(i) $g(a) \in D(a, p, \lambda)$ a.e. in A ; (ii) $s(f) \in S(f, p)$ for all $f \in F$; (iii) $\lambda \in \int_A D(a, p, \lambda)$.*

Proof: To prove (i), take $a \in A$. There is $k^*(a)$ such that for $k > k^*(a)$, $a \in A^k$. Then for $k > k^*(a)$ we have $g^k(a) \in D^k(a, p^{*k}, \lambda^{*k})$. Also, for $k = 1, 2, \dots$

$$p^{*k} g^k(a) \leq p^{*k} e(a) + \sum_F \alpha(a, f) p^{*k} s^{*k}.$$

Taking limits as $k \rightarrow \infty$

$$pg(a) \leq pe(a) + \sum_F \alpha(a, f) ps$$

and therefore $g(a) \in B(a, p)$ a.e in A .

Now take $x \in B(a, p)$. We can have two cases: (a) $px < M(a, p)$ and (b) $px = M(a, p)$.

In case (a), for k sufficiently large

$$p^{*k} x < M^k(a, p^{*k})$$

(since $M^k(a, p^{*k}) \rightarrow M(a, p)$ as $k \rightarrow \infty$). Since $g^k(a) \in D^k(a, p^{*k}, \lambda^{*k})$, then

$$u(a, g^k(a), \lambda^{*k}) \geq u(a, x, \lambda^{*k})$$

Therefore by continuity of u we have that

$$u(a, g(a), \lambda) \geq u(a, x, \lambda)$$

which is to say $g(a) \in D(a, p, \lambda)$. In case (b), take $a \in A/A^p$. Then we can find a sequence $x^i \rightarrow x$ such that $px^i < M(a, p)$. This implies that for k sufficiently large

$$p^k x^i < M^k(a, p^k)$$

which implies

$$u(a, g^k(a), \lambda^{*k}) \geq u(a, x^i, \lambda^{*k})$$

Taking limits first with respect to k and then with respect to i , continuity of u gives

$$u(a, g(a), \lambda) \geq u(a, x, \lambda)$$

and then $g(a) \in D(a, p, \lambda)$. Therefore the proof of (i) is complete, since it was established above that A_p is a set of measure zero when the distribution of income is dispersed.

Now we show (ii). Take $y \in Y^f$. For k sufficiently large $y \in Y^{f,k}$. For $k = 1, 2, \dots$, we have $s^{*k}(f) \in S^k(f, p^k)$ and then for k such that $y \in Y^f$ we have $p^k s^{*k}(f) \geq p^k y$. Taking limits as $k \rightarrow \infty$ this gives

$$ps(f) \geq py$$

which is (ii).

To show (iii), note that $\int_A g^k = \lambda^{*k} \rightarrow \int_A g = \lambda$ as $k \rightarrow \infty$. But we have shown in (ii) that $g(a) \in D(a, p, \lambda)$. Therefore $\lambda = \int_A g(a) \in \int_A D(a, p, \lambda)$, which is the definition of λ being consistent. Q.E.D.

5 Conclusion

In the introduction it was noted that there has been relatively little work on NE in the General Equilibrium literature. The model presented above is an attempt to fulfill this gap. NE are usually bundled with indivisibilities, which makes existence of competitive equilibrium non-trivial. The main contribution of the current research is to prove existence of equilibrium in that situation.

In the paper we have used a key assumption, namely that the distribution of income in the population is dispersed. This assumption allows us to disregard households with budget sets containing troublesome points, i.e. households for whom we could observe discontinuous demand behavior. That makes aggregate behavior smooth, in the sense that aggregate demand changes smoothly with prices. This feature of the model is essential for the existence result.

As it is the case with any consumption externalities, equilibrium cannot be expected to be efficient. The reason is of course the lack of a market for the externalities themselves. In the current setting, inefficiencies could involve the market to be dominated by an inferior standard, compared to alternative standards in the market. A classical example that have been presented in the literature is the home video market, that was dominated by the VHS standard when apparently Betamax was superior.

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