Portfolio Life Cycle Selection – Theory and Experiment

Jens Schubert
Department of Economics
East Carolina University

Under the Supervision of Prof. Jamie Brown Kruse

July 27, 2006

Abstract

In this paper, I compare the results from a laboratory experiment, in which individuals could allocate their funds between a risky asset and a risk-free zero return fund, with stochastic dynamic optimization policies based on expected utility maximization. Specifically, I focus on the predictions based on a negative exponential and concave quadratic utility. In addition, I look at the applicability of a rather exotic functional form to the experimental two security portfolio life cycle decision problem. However, the expected utility theorem has some implications which are not conform with field observations [Mar52], [Kah79], [Rab00], and [Rab01]. Therefore, an alternative theory [Kah79] is critically reviewed that avoids the less arguable implications of expected utility theory [Tve92]. Moreover, possible solutions are derived for a two asset portfolio decision problem in order to show the effects of the different models.

Contents

List of Tables

List of Figures

1 Introduction

Individual's consumption and investment decisions under risk as in the portfolio selection model are subject to many considerations. Clearly, an investor's preferences and their implications are important for the evaluation of investment opportunities (e.g. risky securities). In this paper, I review the general portfolio selection problem, both the two period and the life cycle decision problem. According to the expected utility theory, which is common in the financial economics literature, an investor's decision is characterized by possible payoffs and their respective probabilities: Individuals act as if they ascribe a numerical value (called utility) to each possible outcome and then make choices among alternatives with known odds in order to maximize expected utility. Risk aversion is commonly assumed on part of the economic decision makers (investors). By the expected utility principle, the only explanation for risk aversion is a concave utility function, i.e. lower marginal utility for additional wealth at high wealth levels (*rich person*) compared with low wealth levels (*poor person*). However, in a recent article, the author points out that "for any concave utility function, even very little risk aversion over modest stakes implies an absurd degree of risk aversion over large stakes" [Rab00]. Expected utility theory "says that people will not be averse to risks involving monetary gains and losses that do not alter lifetime wealth enough to affect significantly the marginal utility one derives from that lifetime wealth" [Rab01]. We keep these limitations in mind when analyzing the life cycle portfolio experiment in terms of expected utility theory.

In a life cycle investment problem, decisions made now usually affect future payoffs. In general, a decision made today in order to maximize current wealth without considering future possible outcomes may not automatically be the optimal investment policy for the multi period investment problem. Possible dynamic programming solutions are derived based on different functional utility forms. The findings are applied and compared to a laboratory experiment where individuals could choose between money cash holdings and investment in a risky asset. Liquidity preference theory describes why balances are held in cash rather than in earning assets [Tob58].

Section 2.1 discusses the portfolio selection problem from the perspective of the Expected Return-Variance Rule and the Expected Utility Maxim. The single period decision model is laid out and the finite life cycle decision is reviewed in some detail in section 2.2. The experimental portfolio is described in chapter 3 and moreover, theoretical predictions of different functional forms of the utility function and value function [Kah79] are presented. In chapter 4, I compare the theoretical predictions with the experimental results and show the applicability of the proposed models. The findings are summarized and discussed in the last section.

2 Theoretical Framework

2.1 The Portfolio Selection Problem

In this section, I review the concepts and applicability of different portfolio selection criteria, i.e. the expected returns – variance of returns rule and the expected utility maxim. Moreover, I lay out the intertemporal dynamic programming portfolio selection model introduced by [Sam69] and [Mer69].

2.1.1 Mean–Variance Analysis

The theory of portfolio analysis was first introduced by Markowitz in the early 1950s [Mar52]. The expected returns – variance of returns rule (*EV-Rule*) implies that an investor can "gain expected return by taking on variance, or reduce variance by giving up expected return" [Mar52], which entails that the investor will diversify among different securities. However, likewise important, the EV-Rule takes into account the special case where investment in a single security is the preferred portfolio. In the following, the portfolio selection problem is presented for a range of securities as reported by Markowitz.

In general, a portfolio can be represented by the vector

$$\vec{X} = (X_1, X_2, \ldots, X_\nu, \ldots, X_n)^T, \tag{1}$$

where X_ν denotes the fraction of an investor's total assets invested in the ν^{th} security, i.e. the ratio of amount invested in security ν, w_ν, to total available assets, w. Note that $\sum_{\nu=1}^{n} X_\nu = 1$. For simplicity, assume $X_\nu \geq 0 \ \forall \nu$, i.e. short sales are not allowed. The investor considers the returns, R_ν, of the securities ($\nu = 1, \ldots, n$) to be random variables with expected yield μ_ν. The overall yield of an investor's portfolio can thus be written as

$$R = \sum_{\nu=1}^{n} R_\nu X_\nu \, . \tag{2}$$

The covariance between two securities ν and μ is $\sigma_{\nu\mu}$. Notice the variance of security ν is $\sigma_{\nu\nu} = Var(R_\nu)$. Then, the quadratic form of the variance of the portfolio is

$$Var(R) = \sigma_R^2 = \sum_\nu \sum_\mu X_\nu X_\mu \sigma_{\nu\mu}, \tag{3}$$

which can also be written in matrix notation as $Var(R) = \vec{X}^T \cdot \sigma \cdot \vec{X}$ with variance-covariance matrix σ.

In general, the fractions X_ν are subject to

$$E(R) = \mu_R = \sum_\nu X_\nu \mu_\nu \tag{4}$$

(i.e. the expected return of the portfolio is the weighted sum of the single returns) and m simultaneous linear equations $\sigma \cdot \vec{X} = \lambda \cdot \vec{R}$ [Mar52] (in matrix notation), where λ is the Lagrange multiplier. However, without loss of generality, let $m = 1$, and thus $\sum_\nu X_\nu = 1$ is the only constraint to the optimization problem. According to the EV-Rule, an investor selects among those feasible portfolios that either result in maximum expected return, $E(R)$, holding the variance of the portfolio, $Var(R)$, constant, or those for which $Var(R)$ is minimized, given the expected yield $E(R)$.

The efficient set of portfolios defines all efficient portfolios within the boundary of the attainable set in accordance with the EV-Rule [Mar52]. An efficient set given $E(R)$ ($Var(R)$) is called *isomean line* (*isovariance curve*). The line that combines all points on the isomean lines (i.e. variation of $E(R)$), where $Var(R)$ is minimized (i.e. points at which the isomean lines are tangent to the isovariance curves) is called *critical line* [Mar52].

Generally, an efficient set (also called *dominant combination of assets*) that is a set of assets which minimizes $Var(R)$ for a given $E(R)$, can be found by solving the Lagrangian [Tob58]

$$\sum_\nu \sum_\mu X_\nu X_\mu \sigma_{\nu\mu} - \lambda \left(\sum_\kappa R_\kappa X_\kappa - E(R) \right) =^{min}_{\{\vec{X}\}} . \tag{5}$$

Consider for instance a portfolio with two securities. The model can thus be summarized in the following equations:

$$\begin{aligned} E(R) &= X_1\mu_1 + X_2\mu_2 && (6)\\ Var(R) &= X_1^2\sigma_{11} + X_2^2\sigma_{22} + 2X_1X_2\sigma_{12} && (7)\\ X_1 + X_2 &= 1 && (8)\\ X_{12} &\geq 0 \quad . && (9) \end{aligned}$$

Notice equations 8 and 9 denote the constraints in this two security model. Rewriting equation 9 as $X_2 = 1 - X_1$ and plugging this into equations 6 and 7 allows to express the yield and variance of the portfolio in terms of X_1:

$$\begin{aligned} E(R) &= X_1(\mu_1 - \mu_2) + \mu_{12} && (10)\\ Var(R) &= X_1^2(\sigma_{11} + \sigma_{22} - 2\sigma_{12}) + 2X_1(\sigma_{12} - \sigma_{22}) + \sigma_{22}\ . && (11) \end{aligned}$$

Thus, X_1 and X_2 can be written in terms of the expected returns:

$$X_\nu = (-1)^{\nu-1}\frac{(E(R) - \mu_\nu)}{\mu_1 - \mu_2}, \tag{12}$$

where $\nu = 1, 2$ and $(\mu_1 \neq \mu_2)$. The fractions to be invested in security 1 and 2 can be written as functions of the components of the variance-covariance matrix according to

$$0 = X_\nu^2(\sigma_{11} + \sigma_{22} - 2X_\nu(\sigma_{12} - \sigma_{\nu\nu}) + (\sigma_{\nu\nu} - Var(R)), \tag{13}$$

for $\nu = 1, 2$. An investor acting according to the EV-Rule can choose among combinations of $E(R)$ and $Var(R)$ dependent upon her fixed proportions of assets X_1 and X_2. For instance, solving equation 5 gives the optimal fraction to be invested in security $\nu = 1$ and 2, holding $E(R)$ constant:

$$X_\nu = \frac{(\sigma_{\nu\nu} - \sigma_{12}) \cdot E(R) \cdot R_\nu}{R_1^2(\sigma_{22} - \sigma_{12}) + R_2^2(\sigma_{11} - \sigma_{12})}. \tag{14}$$

The portfolio selection problem (equation 5) can be solved accordingly for $n \geq 2$.

2.1.2 Expected Value Decision Principle

So far, the portfolio selection problem has only been considered under the EV-Rule without looking at the odds of possible asset payoffs. However, portfolio selection is a situation involving risk. Investors choose among so called *lotteries* η (*gambles*) that are described by their payoffs $(R_1, \ldots, R_n)$ and respective probabilities of occurrence $(p_1, \ldots, p_n)$: $\quad \eta = \{(R_1, \ldots, R_n), (p_1, \ldots, p_n)\}$. Let an individual's investment decision be characterized by the determination of the probabilities of possible asset payoffs [Hua88]. Then, by the expected value decision principle, an investor maximizing the expected value of the portfolio, consequently, will always put all available assets in the security with the greatest expected return or will be indifferent between portfolios that only contain securities with same greatest expected yield [Mar91]. The expected value of a portfolio $\vec{X}$ can be expressed as

$$E(\vec{X}) = \sum_{\nu}^{n} p_\nu \sum_{\mu} X_\mu R_\mu^\nu = \sum_{\mu} X_\mu \sum_{\nu}^{n} p_\nu R_\mu^\nu = \sum_{\mu} X_\mu \mu_\mu. \qquad (15)$$

2.1.3 Expected Utility Principle

A common descriptive model of decision making under risk in the financial economics literature is the expected utility principle (*EU-maxim*), which includes the expected value principle as special case. According to the EU-maxim, the investor ascribes a real number (called utility) to each possible outcome, and among risky alternatives chooses the one with the highest expected value of utility [Mar91].

An investor can be represented by her preference relation defined on an aggregation of investment plans (*lotteries*) [Hua88]. Suppose there is a preordering on the set of lotteries that satisfies the following axioms [Ing87]. For any arbitrary lotteries η, ω, and ξ:

Completeness – either $\eta \succeq \omega$ or $\omega \succeq \eta$ or $\eta \sim \omega$, i.e. either η is at least good as ω, or ω is at least good as η, or the investor is indifferent between η and ω.

Reflexivity – for every lottery $\eta \succeq \eta$.

Transitivity – if $\eta \succeq \omega$ and $\omega \succeq \xi$, then $\eta \succeq \xi$, i.e. if η is preferred to ω and ω is preferred to ξ, then η is preferred to ξ.

Continuity – if $\eta \succ \omega$ and $\omega \succ \xi$ then there is a number $\lambda \in [0,1]$ such that $\omega \sim \lambda \cdot \eta + (1-\lambda) \cdot \xi$. Note λ is unique unless $\eta \sim \xi$.

Independence [1] – let $\eta = \{(R_1, \ldots, R_\nu, \ldots, R_n), (p_1, \ldots, p_n)\}$ and $\omega = \{(R_1, \ldots, \xi, \ldots, R_n), (p_1, \ldots, p_n)\}$ then $\eta \sim \omega$ if $R_\nu \sim \xi$. If $\xi = \{(R_1^\nu, \ldots, R_m^\nu), (p_1^\nu, \ldots, p_m^\nu)\}$ is another lottery, then $\eta \sim \omega \sim \{(R_1, \ldots, R_{\nu-1}, R_1^\nu, \ldots, R_m^\nu, R_{\nu+1}, \ldots, R_n), (p_1, \ldots, p_{\nu-1}, p_1^\nu, \ldots, p_m^\nu, p_{\nu+1}, \ldots, p_n)\}$.

Dominance – let $\eta = \{(R_1, R_2), (p_1, 1-p_1)\}$ and $\omega = \{(R_1, R_2), (p_1, 1-p_2)\}$, then given $R_1 > R_2$: η is strictly preferred to ω ($\eta \succ \omega$) if and only if $p_1 > p_2$.

A preference relation has an expected utility representation if the above axioms are satisfied. Moreover, a so called *von Neumann-Morgenstern* utility function $u(\eta)$ (*VNM*) exists with the expected utility property representing the investor's risk preferences over different lotteries [Jeh01]. Lottery η is preferred over lottery ω if and only if the expected utility of payoff vector $\vec{R}$ of η exceeds the expected utility of payoff vector $\vec{R}'$ of ω [Hua88]:

$$E[u(\vec{R})] \geq E[u(\vec{R}')], \tag{16}$$

where the expected utility $EU(\eta) = E[u(\vec{R})]$ can be written as

$$EU(\eta) = \sum_\nu p_\nu u_\nu = \sum_\nu p_\nu u \left(\sum_\kappa X_\kappa R_\nu^\kappa \right), \tag{17}$$

[1] Notice that the independence axiom is often violated in empirical experiments [Hua88].

i.e. the utilities of likely outcomes are weighted by their probabilities. In other words, based on a set of axioms of preference ordering including the independence axiom, the expected utility principle describes decision making under risk, which implies that a VNM expected utility function represents the ordering that is linear in probabilities. The investment decision is thus dependent upon the functional forms of the VNM utility (see section 2.3).

Back to the portfolio selection problem, in general for N states and M assets in the portfolio, the investor will choose X_ν in order to maximize expected utility of wealth w [Jeh01]:

$$\underset{\{\vec{X}\}}{max}\; EU = \underset{\{\vec{X}\}}{max} \sum_{\nu}^{N} p_\nu \cdot \sum_{\kappa}^{M} u\left(w + X_\kappa R_\nu^\kappa\right) \tag{18}$$

That is, the investor will decide to invest X_κ in security κ in order to reach the highest indifference curve attainable, given her feasible set of opportunities. From equation 18 it becomes apparent that utility is ascribed to final states of wealth rather than to changes in wealth.

2.2 The Intertemporal Model

So far, only single period models have been examined. In case of the life cycle portfolio decision problem, however, investors make decisions concerning consumption and investment over many periods. The underlying principle is that investment decision at time τ implies a wealth transfer from period τ to the next period $\tau + 1$ by forming a portfolio: At each period τ, the investors invest their available funds (after consumption is made) in available assets, in order to sell these the next period to make their consumption and subsequent investment decisions at $\tau + 1$ [Ing87].

For the sake of simplicity, the date of death T is assumed to be known. Moreover, there is no reason for bequest after T. The investor is now concerned not only about her current consumption but also about consumption in future periods. Let's make the assumption that the utility function of wealth is additively separable [Ing87]:

$$u(C_1, \ldots, C_{T-1}) = \sum_{\tau=0}^{T-1} u(C_\tau), \tag{19}$$

where $u(C_\tau)$ is the concave VNM utility function of consumption C_τ. The latter is a function of wealth at period τ, w_τ, and the change in wealth over time:

$$C_\tau = w_\tau - \frac{w_{\tau+1}}{1+r}, \tag{20}$$

with the exogeneous rate of return r. According to the EU-maxim, an investor chooses a consumption stream to maximize her expected value of total utility over time with respect to consumption C_τ and wealth w_τ [Sam69]:

$$\underset{\{C_\tau, X_\tau\}}{max} E[u\left(C_0, \ldots, C_{T-1}\right)] = \underset{\{C_\tau, X_\tau\}}{max} \sum_{\tau=0}^{T-1} (1+\rho)^{-\tau} u\left(w_\tau - \frac{w_{\tau+1}}{1+r}\right), \tag{21}$$

where ρ is the discount rate. The first order conditions of a regular interior maximum are derived by partially differentiating equation 21 with respect to w_τ and setting the result equal to zero [Sam69]:

$$0 = (1+\rho)^{-\tau} u' \left(w_\tau - \frac{w_{\tau+1}}{1+r}\right) - \frac{(1+\rho)^{1-\tau}}{(1+r)} u' \left(w_{\tau-1} - \frac{w_\tau}{1+r}\right). \tag{22}$$

The first order conditions are recursive:

$$u' \left(w_\tau - \frac{w_{\tau+1}}{1+r}\right) = \frac{1+\rho}{1+r} \cdot u' \left(w_{\tau-1} - \frac{w_\tau}{1+r}\right). \tag{23}$$

Given the boundary conditions, $(w_0, \ w_T)$, i.e. initial funds and prescribed wealth at T, equation 23 can be solved for concave utility functions.

Let me now lay out the portfolio decision problem over time with two assets, i.e. a risky one with uncertain yield R_τ and a safe security with known return r. The return to the former is assumed randomly distributed and independent of returns of previous periods. Recall from the single period model that an investor decides to assign $X_\tau \cdot w_\tau$ of her funds in the risky asset and $(1 - X_\tau) \cdot w_\tau$ in the safe security. Hence, the expected utility maximization problem can be formulated in the problem's associated maximum value function $M_{T-1}(w_0)$ as

$$\begin{aligned} M_{T-1}(w_0) &= \max_{\{C_\tau, X_\tau\}} E\left[\sum_{\tau=0}^{T-1} (1+\rho)^{-\tau} u(C_\tau)\right] \text{ with} && (24) \\ C_\tau &= w_\tau - \frac{w_{\tau+1}}{(1+r)(1-X_\tau) + X_\tau R_\tau} \quad . && (25) \end{aligned}$$

Being uncertain about the outcome of return R_t, the investor chooses C_t and X_t at period t. However, today's actions are based on the fact that at future periods $\tau > t$, the investor has full knowledge about the yields R_τ and thereby wealth w_τ of previous periods $\tau \in [0,\ t]$. Allocation decisions C_τ and X_τ are made sequentially at each period $\tau = 0,\ 1,\ \ldots,\ T-2$. Thus, at the end of the planning period (i.e. last allocation decision in a life cycle portfolio framework), the investor faces the one-period problem

$$M_{T-1}(w_{T-2}) = \max_{\{C_{\tau-2}, X_{\tau-2}\}} \left\{ u(C_{T-2}) + E\left[\frac{u(C_{T-1})}{1+\rho}\right] \right\}, \qquad (26)$$

where $C_{T-1} = w_{T-1}$ based on the previous assumption of no bequest of wealth at death, i.e. $w_T = 0$:

$$C_{T-1} = w_{T-1} = (w_{T-2} - C_{T-2})\left[(1+r)(1-X_{T-2}) + X_{T-2}R_{T-2}\right]. \qquad (27)$$

Next, partial differentiation of equation 26 with respect to C_{T-2} and X_{T-2} and solving the first order conditions simultaneously

$$\begin{aligned} \frac{\partial M_{T-1}}{\partial C_{T-2}} &= \frac{\partial u(C_{T-2})}{\partial C_{T-2}} \\ &- (1+\rho)^{-1} E\Big[u'(C_{T-1})\left[(1+r)(1-X_{T-2}) + X_{T-2}R_{T-2}\right]\Big] && (28) \\ \frac{\partial M_{T-1}}{\partial X_{T-2}} &= E\left[u'(C_{T-1})\,(w_{T-2} - C_{T-2})\,(R_{T-2} - 1 - r)\right] && (29) \end{aligned}$$

gives the optimal decisions (C^*_{T-2}, X^*_{T-2}) as functions of initial wealth w_{T-2} alone [Sam69]. One can derive the maximum value function M_{T-1} explicitly by plugging in C^*_{T-2} and X^*_{T-2}. By the envelope theorem, which states that the total effect on the optimized value of the objective function can be derived by partial differentiation of the utility function [Jeh01], one can relate the derivatives of u to those of the maximum value function:

$$M_{T-1}(w_{T-2})' = u'(C_{T-2}). \tag{30}$$

With the knowledge of $M_{T-1}(w_{T-2})$ one can easily derive the optimal decisions one period earlier:

$$M_{T-2}(w_{T-3}) =_{\{C_{T-3},X_{T-3}\}}^{max} \left\{ u(C_{T-3}) + \frac{M_0(w_{T-2})}{1+\rho} \right\}, \tag{31}$$

where $w_{T-2} = (w_{T-3} - C_{T-3})\big[(1+r)(1-X_{T-3}) + X_{T-3}R_{T-3}\big]$ (equation 25). Partially differentiating and simultaneous solving of the first order conditions gives the optimal decisions (C^*_{T-3}, X^*_{T-3}) and thereby $M_{T-2}(w_{T-3})$. Proceeding recursively in an analogous manner for $T-4, \ldots, 0$, the stochastic programming problem is being solved.

2.3 Behavior Under Risk

Let me now relate the expected utility hypothesis to its implied risk preferences. The expected utility of participating in an actuarially fair lottery[2] η, $E[u(w_0+\eta)]$ (17), and the utility of non-participation, $u\left(E[w_0+\eta]\right)$, can be used to describe the relationship between an investor's VNM utility function and her risk attitudes. In general, an individual is said to be risk averse, if she shuns or is indifferent to any actuarially fair lottery. Strict risk aversion,

[2]An actuarially fair lottery is defined as one with expected value of the gamble equal to zero, $E(\eta) = 0$, in other words when its expected payoff is zero.

refers to an individual who is unwilling to accept any actuarially fair lottery [Hua88]. Put differently, *risk averse* investors would rather receive the expected value of η with certainty than facing the risky outcome [Jeh01]:

$$u\big(E[w_0+\eta]\big) > E\big[u(w_0+\eta)\big]. \tag{32}$$

An investor is (strictly) risk averse if and only if her VNM utility functions of wealth $u(w)$ is (strictly) concave: She would prefer wealth level w_0 with certainty over an actuarially fair lottery η of either rising to w^+ or falling to w_-, if a straight line drawn from $u(w_-)$ to $u(w_+)$ passes below $u(w_0)$[3]. In addition, her marginal utility of outcome of the lottery (return), $u'(R)$, is a declining function of R. Thus, the indifference curves of a risk-averse individual are concave upwards. An investor with a strictly increasing, concave utility function will only invest in a risky security if the risk premium of the risky asset, $E[\hat{R}-r]$[4], is strictly positive [Hua88].

On the other hand, a *risk-lover* is characterized by a marginal utility of return which is an increasing function of R, and thus the indifference map of a risk loving investor is a set of concave downwards curves. The uncertain outcome of the lottery over the sure thing is preferred by a risk lover:

$$E\big[u(w_0+\eta)\big] > u\big(E[w_0+\eta]\big). \tag{33}$$

A risk loving investor would prefer the lottery over the sure thing if the line $\overline{u(w_-)u(w_+)}$ would pass above the utility of the current wealth level, $u(w_0)$.

Contrary, a *risk neutral* investor is indifferent between the expected yield of the portfolio with certainty and the risky outcome:

$$E\big[u(w_0+\eta)\big] = u\big(E[w_0+\eta]\big) = u\big(E[w_0]\big)^5. \tag{34}$$

[3]Notice that the subscripts $-$, 0, and $+$ are chosen to emphasize loss, current wealth, and gain, respectively.

[4]with $\hat{R}$ being the random yield of the risky and r the certain interest rate of the riskless alternative, respectively.

[5]Note $E(\eta) = 0$.

She is indifferent between $w_0 + \eta$ and w_0 if $\overline{u(w_-)u(w_+)}$ coincides with her VNM utility function $u(w)$ at all wealth levels $w \in [w_-, w_+]$.

2.3.1 Measures of Risk Aversion

A utility function is concave, representing risk-averse choices if it is twice differentiable and most importantly, if and only if $u''(w) < 0$ [Ing87]. In other words, since risk aversion means that the certainty equivalent w_0 is smaller than the expected value of the lottery $E[\eta]$, the VNM utility function of a risk averse investor must be concave. Put differently, risk aversion is equivalent to the concavity of the VNM utility function.

The so called *Arrow-Pratt measure* [Pra64]:

$$r(w) = -\frac{u''(w)}{u'(w)} \tag{35}$$

is a measure of the curvature of an investor's utility function and can be interpreted in different ways as a measure of local risk aversion [Pra64]. Likewise, a local measure of aversion to risks as a proportion of funds is the Arrow-Pratt measure of relative risk aversion:

$$\tilde{r} = w \cdot r(w). \tag{36}$$

An investor's utility function indicates decreasing, increasing, or constant absolute (relative) risk aversion, when $r(w)$ $(\tilde{r})$ is a strictly decreasing, increasing, or constant function, respectively. The utility functions used in this work display constant (increasing) absolute (relative) or decreasing (constant) absolute (relative) risk aversion. Decreasing absolute risk aversion (DARA), $dr(w)/dw < 0 \, \forall \, w$, implies that the investor's demand for the risky security increases with wealth. Likewise, under constant relative risk aversion (CRRA), $d\tilde{r}/dw = 0 \, \forall \, w$, the share of wealth invested in the risky asset is a constant, $dX_R/dw = 0$. On the other hand, constant absolute risk aversion

(CARA), $dr(w)/dw = 0 \, \forall \, w$, denotes unchanged demand for the risky security with respect to initial wealth. Increasing relative risk aversion (IRRA), $d\tilde{r}/dw > 0$, indicates that the proportion invested in the risky asset declines as wealth increases, $dX_R/dw < 0$ (see for instance [Hua88] for proof).

3 Experimental Aspects

In this section, I describe the experimental design and develop investment solutions for the allocation problem – both for the different single period approaches and for the stochastic dynamic optimization in general.

3.1 Description of the Portfolio

Consider a two-asset portfolio allocation problem, one asset with a risky return and the other with zero return for sure (i.e. cash fund). Given an initial amount to invest, the investor must decide what proportions of her assets to put in the risky asset and what proportion to hold in cash. Put another way, the subjects can build different portfolios by holding the proportion $(1 - X_R)$ of their funds in cash with zero yield and zero variance and by investing X_R in a volatile security, which can either yield a good ($R_g > 0$) or a bad ($R_b < 0$) return. The risky security has a five percent mean return and non-zero variance. Negative proportions of X_R are excluded by definition (no short sales). The outcome of the risky security is determined by chance (i.e. toss of a coin). There were three versions of the risky security, i.e. low, medium, and high risk (see Table 1).

	R_g	R_b	$Var(R)$	$sd(R)$
Low	0.2	-0.1	0.0225	0.15
Medium	0.6	-0.5	0.3025	0.55
High	1.0	-0.9	0.9025	0.95

Table 1: Versions of the risky security with expected yield, $\mu(R) = 0.05$

3.2 Experimental Design

Subjects participating in the experiment were informed that at each of 33 periods they would receive 25 lab dollars. The reimbursement rate for participation was 1 to 100 lab dollars at the end of the session. Participants had to decide how to allocate their available lab dollars between the cash fund and the risky asset. They were told that at each period the outcome of the risky asset would be determined by the flip of a coin. Moreover, at each of the 33 periods, before participants made their investment decisions, they were informed whether the alternative to cash holdings is the low, medium, or high risk security. In any case, they were uncertain about the future rate of return of the risky asset. Thus, the proportion of available funds put into the risky asset involved a risk of capital gain or loss. However, importantly, participating individuals had to make a decision first and thereby had to fix their portfolios for each decision period. The coin was flipped for the whole group participating in a session rather than individually, and thus the outcome of each period was announced after all individuals had made their decisions.

3.3 The Allocation Problem

A possible source of liquidity preference is the investor's uncertainty about the future of interest rates [Tob58]. The investment of funds in risky assets involves a risk of capital gain or loss. The risk that comes along with the decision to invest in the risky security is measured by the standard deviation of R, $\sigma_R = \sqrt{Var(R)}$. The standard deviation measures the spread of possible returns of the security around the mean value μ_R. Thus, a high standard deviation means a high chance of relatively large capital gains but also a high chance of relatively large capital losses. Likewise, a risky asset with low standard deviation around its expected yield decreases the chance of high gains (losses) (i.e. small likelihood of large deviations from the mean). Note that zero standard deviation would give μ_R for sure and $E\left[u(\vec{R})\right] = u\left(\mu_R\right)$.

Let us assume that subjects are expected utility maximizers. Then, if the expected utility of the certain outcome (zero return of cash holdings) is greater than the uncertain asset allocation, an investor acting in accordance with the EU-maxim will hold all funds in cash. The single period expected utility maximization problem (equation 18) can be written as

$$\underset{\{\vec{X}\}}{max}\ EU(\eta) = \underset{\{X_R\}}{max} \sum_{\nu=1}^{N} p_v \cdot u(w(1 + X_R R_\nu)) \tag{37}$$

with $M = 2$ and $R_2 = r = 0$ (cash). As noted earlier, the volatile security can yield two outcomes. Thus, we can rewrite equation 37:

$$\underset{\{X_R\}}{max}\ EU(\eta) = \underset{\{X_R\}}{max} \left\{ p_g \cdot u\big(w(1 + X_R R_g)\big) + p_b \cdot u\big(w(1 - X_R\,|R_b|)\big) \right\}. \tag{38}$$

The first and second order conditions for a maximum are

$$EU' = p_g u'\big(w(1 + X_R R_g)\big) R_g - p_b u'\big(w(1 - X_R |R_b|)\big)|R_b| = 0 \tag{39}$$

$$EU'' = p_g u''\big(w(1 + X_R R_g)\big) R_g^2 + p_b u''\big(w(1 - X_R |R_b|)\big) R_b^2 < 0. \tag{40}$$

Concavity of the utility function ($u''(w) < 0$) satisfies the second order condition. With these results, the expected utility maximization problem can be solved in a straight forward manner by applying different functional forms reflecting risk averse behavior.

Notice, an extension of the model with more than two securities can be found in [Ing87] and [Hua88], for instance.

3.3.1 Power Utility Function

First, consider a power utility function of the form

$$u(w) = \frac{w^{1-\alpha}}{1-\alpha}, \qquad \alpha \in (0,\ 1),^{6} \tag{41}$$

[6] see Figure 1: grey line refers to $u(w)$ with $\alpha = 0.5$

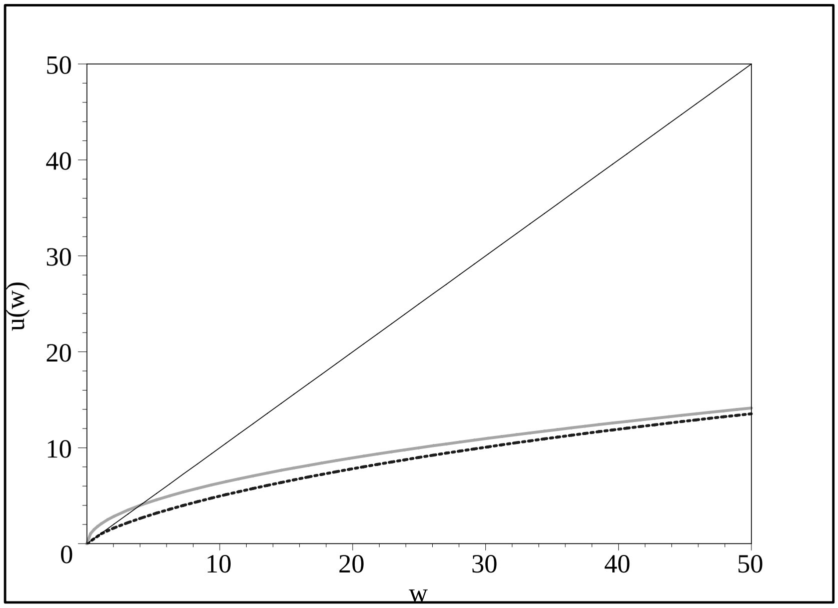


Figure 1: Power- (solid grey) and Expo-Power-Utility (dashed black)

which exhibits CRRA, $\tilde{r} = \alpha$ (36) and DARA, $r(w) = \alpha/w$ (35). Substituting (41) into equation 38 and solving for X_R gives the optimal fraction to be invested in the risky security,

$$X_R^* = \frac{\sqrt[\alpha]{p_g R_g} - \sqrt[\alpha]{p_b |R_b|}}{R_g \sqrt[\alpha]{p_b |R_b|} + |R_b| \sqrt[\alpha]{p_g R_g}} \,. \tag{42}$$

Notice, importantly, that the optimal fraction, X_R^*, is a function of the yields R_g and R_b alone and thus independent of initial wealth. In a life cycle investment problem, the investor will always invest the same fraction of his funds in the risky asset, i.e. at each decision period, the optimal share of wealth is independent of initial wealth for CRRA utility functions [Sam69].

Recall from section 2.1.3 that in expected utility theory, value is ascribed to final states rather than to mere changes in wealth. In our lab experiment, $p_g = p_b = 0.5$ (i.e. toss of a coin). Thus equation 42 can be written as

$$X_R^* = \frac{\sqrt[\alpha]{R_g} - \sqrt[\alpha]{|R_b|}}{R_g \sqrt[\alpha]{|R_b|} + |R_b| \sqrt[\alpha]{R_g}} \,. \tag{43}$$

If $X_R^* > 1$ then $X_R^* := 1$, that is all funds will be put in the risky security.

3.3.2 Negative Exponential Utility Function

The negative exponential utility function

$$u(w) = -\frac{e^{-\eta w}}{\eta}, \quad \eta > 0 \tag{44}$$

displays CARA, $r(w) = \eta$, and IRRA, $\tilde{r} = \eta w$. Importantly, the optimal share of wealth invested in the risky asset

$$X_R^* = \frac{ln(R_g) - ln(|R_b|)}{\eta w(R_g + |R_b|)} \tag{45}$$

is a function of initial wealth, contrary to the optimal proportion of a power utility function (equation 43). With rising wealth, the share invested in the risky security decreases, and the individual invests more and more in the risk-free asset (holds her money in cash).

Notice that the negative exponential utility function (44) is bounded from above, that is infinite wealth has a finite utility, $\lim_{w\to\infty} u(w) = 0$.

3.3.3 Quadratic Utility Function

Next, consider a concave quadratic utility of the form

$$u(w) = w - \frac{bw^2}{2}, \quad b > 0, \tag{46}$$

which exhibits increasing absolute risk aversion (IARA), $r(w) = \frac{b}{1-bw}$, and IRRA, $\tilde{r} = \frac{bw}{1-bw}$, $\tilde{r}' = \frac{b}{(1-bw)^2} > 0$. IARA is one of the flipsides of concave quadratic utility functions. The second drawback is that they are decreasing after a certain point. As a result, one must be very cautious in choosing b to assure that all eventual outcomes remain in the range below the functions' maximum, $w < 1/b$. However, it can be shown that expected utility only depends on the mean and variance of return and is therefore consistent with mean-variance analysis:

$$E\big[u(w)\big] = w(1 + X\mu_R) - \frac{bw(1 + X\mu_R)^2}{2} - \frac{bwX^2\sigma_R}{2}\,. \tag{47}$$

For risk averse investors, only minimum variance portfolios are held as long as outcomes are in the range of increasing utility ($w < 1/b$) [Ing87].

The share to be optimally invested in the risky asset,

$$X_R^* = \frac{R_g(1 - bw) - |R_b|(1 - bw)}{bw\big(R_g^2 + |R_b|^2\big)}\,, \tag{48}$$

depends on available funds, w.

3.3.4 Expo-Power Utility Function

In a recent paper, Holt and Laury showed that a so called *hybrid expo-power* utility function of the form

$$u(w) = \frac{1 - exp\left\{-\beta w^{1-\gamma}\right\}}{\beta} \tag{49}$$

fits most of their data from lottery choice experiments quite closely [Hol02], and moreover, avoids the absurd predictions which they received from a negative exponential utility function (CARA). The measures of absolute and relative risk aversion are

$$r(w) = \frac{\gamma + \beta(1 - \gamma)w^{1-\gamma}}{w} \quad \text{and} \tag{50}$$

$$\tilde{r}(w) = \gamma + \beta(1 - \gamma)w^{1-\gamma}\,, \tag{51}$$

respectively. Thus, the expo-power utility function exhibits decreasing, constant, or increasing absolute and decreasing or increasing relative risk aversion depending on different parameter values, in order to model several risk-preference structures [Sah93].

The dashed black line in Figure 1 refers to an expo-power utility with parameter values $\beta = 0.03$ and $\gamma = 0.27$. This expo-power utility function exhibits DARA and IRRA. At low wealth levels, it is not as steep as the power utility function $u(w) = \frac{w^{0.5}}{0.5}$ (solid grey line); however, it exhibits higher marginal utility at high wealth levels. Based on the expo-power utility function (49), the expected utility maximization problem can neither be solved algebraically nor numerically for the optimal proportion X_R. Thus, the expo-power utility cannot serve as a descriptive risk averse utility function in order to solve the stochastic dynamic optimization problem.

3.3.5 Cumulative Prospect Theory Value Function

An alternative to the expected theory model, called *prospect theory*, was introduced in the late 1970s by Kahneman et al. [Kah79]. In prospect theory, contrary to expected utility theory, "value is assigned to gains and losses rather than to final assets" [Kah79]. Moreover, probabilities are replaced by so called *decision weights*, $\pi(p)$[7]. Individuals acting according to prospect theory, choose the prospect of highest value $V(\eta)$ [Kah79]:

$$V(\eta) = \sum_{\nu} \pi(p_\nu) v(R_\nu). \tag{52}$$

In terms of the experimental portfolio selection problem, this can be formulated as

$$\begin{aligned} \overset{max}{\{X_R\}}\, V(\eta) &= \overset{max}{\{X_R\}}\, V(R_g, p_g; R_b, p_b) \\ &= \overset{max}{\{X_R\}} \left\{ \pi(p_g) \cdot v(wX_R R_g) + \pi(p_b) \cdot v(-wX_R |R_b|) \right\}. \end{aligned} \tag{53}$$

Prospect theory is thus a generalization of the expected utility theory since it relaxes the expectation principle [Kah79] (see equation 38). Like the utility function, the function $v(x)$ ascribes numbers to every outcome x. The

[7]Decision weights are likely smaller than probabilities.

underlying hypothesis is that the value function is concave above the current wealth level (reference point) and convex below the reference point, which reflects decreasing marginal value both for gains and losses. This is equivalent to risk aversion in the domain of gains with concomitant risk seeking behavior in the domain of losses. Indeed, Tversky et al. [Tve92] propose a *cumulative prospect theory value function* of the form

$$v(x) = \begin{cases} x^{\tau} : & x \geq 0 \quad \text{(gain)} \\ -\kappa \cdot |x|^{\tau} : & x < 0 \quad \text{(loss)} \end{cases} . \tag{54}$$

Their parameter estimates $\tau = 0.88$ and $\kappa = 2.25$ [Tve92] refer to a value function that is concave in the positive domain, convex in the negative domain, and roughly twice as steep for losses as for gains.

Solving the value maximization problem (53)[8] gives the optimal fraction to be invested in the risky asset: Either $X_R^* = 0$ (trivial solution) if $R_g < \left(\kappa \frac{\pi(p_b)}{\pi(p_g)}\right)^{\frac{1}{\tau}} |R_b|$ or $X_R^* = 1$ if $R_g > \left(\kappa \frac{\pi(p_b)}{\pi(p_g)}\right)^{\frac{1}{\tau}} |R_b|$, respectively. Hence, diversification is not captured by the cumulative prospect theory value function. The investor will be indifferent between cash holdings and investment in the risky security if and only if $R_g = \left(\kappa \frac{\pi(p_b)}{\pi(p_g)}\right)^{\frac{1}{\tau}} |R_b|$.

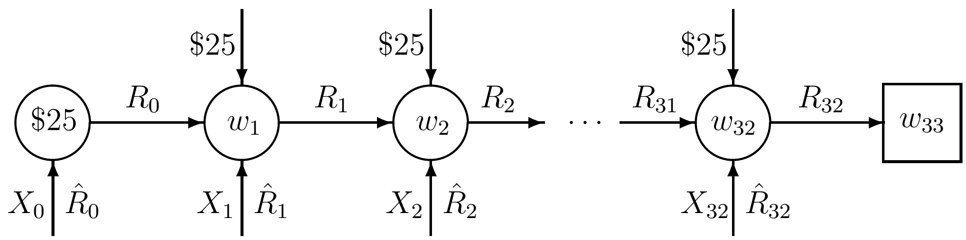
Again, like the expo-power utility, the cumulative prospect theory value function is insufficient to adequately solve the stochastic dynamic optimization problem.

3.4 Stochastic Dynamic Optimization

Recall from section (2.2) that at each period τ, the investor makes consumption and investment decisions without knowing how the rate of return of his portfolio will turn out. Hence, a decision made at τ affects available funds and respective payoffs in the future. Our investment experiment is a discrete sequential optimization problem. Investment decisions are made at certain

[8] Notice that $\pi(p_g) \equiv \pi(p_b) \leftrightarrow p_g \equiv p_b$.

Figure 2: Dynamic Flowchart of the Experiment



times (*stages*). One can think of the experimental decision problem graphically as shown in Figure 2. The individuals can only assign their funds w_τ to the risky security or hold them in cash, consumption is zero, $C_\tau = 0$ for $\tau = 0, \ldots, 32$, except at the end of the life-cycle experiment where terminal wealth equals terminal consumption, $w_{33} = C_{33}$. At each stage, the investor gets 25 lab dollars in virtual income and decides how much of her current available funds to put in the risky security, $X_\tau^R w_\tau$, and how much to keep in cash, $(1 - X_\tau^R)w_\tau$. The participating individuals know, however, that the yield of the risky security, $\hat{R}_\tau$, is a random effect determined by the toss of a coin. At the next period (i.e. $\tau + 1$), importantly, the individuals have the knowledge how the rate of return of the risky security, R_τ, turned out in period τ. In general, wealth (available funds), $w_{\tau+1}$, is a function of the state variable w_τ, the control parameter X_τ^R, and the uncontrollable exogenous rate of return R_τ:

$$\begin{aligned} w_{\tau+1} &= f(w_\tau, X_\tau^R, R_\tau) \\ &= 25 + w_\tau\big(1 + X_\tau^R R_\tau\big), \ \tau = 0,\ 1,\ \ldots,\ 32 \text{ [9]}. \end{aligned} \tag{55}$$

In order to solve the dynamic optimization problem, we first go to the end of the life cycle decision problem to find the optimal investment decision at stage 32, where the first order condition is (see equations 26 through 29)

$$\begin{aligned}\frac{\partial M_{33}}{\partial X_{32}^{R}} &= 0 + (1+\rho)^{-1} E\left[\frac{\partial u}{\partial w_{33}} \cdot \frac{\partial w_{33}}{\partial X_{32}^{R}}\right] \\ &= E\left[u^{'}(w_{33}) \cdot w_{32} R_{32}\right] = 0\,, \end{aligned} \tag{56}$$

which is equivalent to

$$0 = p_g \cdot u^{'}(w_{33}) \cdot w_{32} R_{32}^{g} - p_b \cdot u^{'}(w_{33}) \cdot w_{32} |R_{32}^{b}|\,. \tag{57}$$

Solving for $X_{32}^{R}(w_{32})$ and plugging the result into the maximum value function gives M_{33} in terms of w_{32} and R_{32} alone. Generally, the maximum value function at earlier stages is

$$M_{\tau}(w_{\tau}) =_{\{X_{\tau}\}}^{max} \left\{u(w_{\tau}) + (1+\rho)^{-1} E\big[M_{\tau+1}(w_{\tau+1})\big]\right\}. \tag{58}$$

Again, differentiating with respect to X_{τ},

$$0 = (1+\rho)^{-1} E\left[M_{\tau+1}^{'}(w_{\tau+1}) \cdot w_{\tau} R_{\tau}\right], \tau = 31,\ \ldots,\ 1,\ 0\,, \tag{59}$$

and subsequent solving for X_{τ}^{*} gives M_{τ} explicitly. We have thus found the optimal investment policy for each stage, X_{τ}^{*}, as a function of flowing wealth. Notice, since the only objective is to maximize terminal wealth, the discount factor, $(1+\rho)$, cancels out and therefore does not influence investment decisions at any stage (see equation 59).

The results of the stochastic dynamic programming were obtained using Maple® 8 (see appendix A). The second (last) column in Table 2 shows the optimal investment policy for an expected utility maximizer with a risk averse negative exponential (concave quadratic) utility. The results in Table 2 imply that the optimal decision at any stage τ is a function of available funds w_{τ} (see equation 55) and returns of the risky security at τ, both for the negative exponential and concave quadratic utility. Hence, whatever the

τ	X^*_τ (Neg. Expo.)	X^*_τ (Quadratic)
0	$\frac{ln(R^g_0)-ln(\lvert R^b_0\rvert)}{\eta w_0(R^g_0+\lvert R^b_0\rvert)}$	$\frac{R^g_0(1-bw_0)-\lvert R^b_0\rvert(1-bw_0)}{bw_0\left({R^g_0}^2+\lvert R^b_0\rvert^2\right)}$
$\vdots$	$\vdots$	$\vdots$
t	$\frac{ln(R^g_t)-ln(\lvert R^b_t\rvert)}{\eta w_t(R^g_t+\lvert R^b_t\rvert)}$	$\frac{R^g_t(1-bw_t)-\lvert R^b_t\rvert(1-bw_t)}{bw_t\left({R^g_t}^2+\lvert R^b_t\rvert^2\right)}$
$\vdots$	$\vdots$	$\vdots$
32	$\frac{ln(R^g_{32})-ln(\lvert R^b_{32}\rvert)}{\eta w_{32}(R^g_{32}+\lvert R^b_{32}\rvert)}$	$\frac{R^g_{32}(1-bw_{32})-\lvert R^b_{32}\rvert(1-bw_{32})}{bw_{32}\left({R^g_{32}}^2+\lvert R^b_{32}\rvert^2\right)}$

Table 2: Optimal Dynamic Investment Decisions

previous decisions and states are, all subsequent decisions must constitute an optimal investment strategy with regard to the states resulting from earlier decisions ($< \tau$) [Par00]. At each state ($\tau > 0$), the investor knows the return of the risky security one period earlier with certainty and will choose how much to invest optimally at stage τ.

4 Results

In three sessions, altogether 27 students from upper division undergraduate economics courses participated in the experiment. Prior to the experiment, the sequence of low, medium, and high risk securities (eleven each) was randomly decided for all sessions together. In each session, for each period, the random outcome of the risky security was determined for the whole group rather than individually. Moreover, each group was previously split into "heads" and "tails" to control for bias due to a streak of good (bad) draws – for each period, nearly half of each group received a good or bad return dependent upon whether the toss gave head or tail, respectively. Thus, there are three different sessions with two perfectly negative correlated strings (see Table 7 in Appendix B). Since dynamic optimal investment strategies are affected by initial wealth at each stage, the aggregate behavior of these six strings will be analyzed separately.

Figure 3 (4), Figure 5 (6), and Figure 7 (8) in Appendix B show the average total funds and average amounts invested in the risky security for the first, second, and third session, respectively[10]. Not investing any funds at all in the risky security would result in 825 lab dollars for sure (denoted by the light grey line (Income)). On average, half of the students in each group made less (more) than 825 lab dollars (the strings in the first column of each session in Table 7 refer to the groups that did worse and vice versa). The plots in Figures 3 through 8 indicate that on average, participating subjects became more risk averse after they reached a certain level of wealth (toward the end of the 33 period experimental life cycle). In addition, the plots show that on aggregate, the share invested in the low risk asset is bigger than the medium than the high risk version. The mean of all investment shares over time is $\overline{X}_\tau = 0.435(0.337)$, ranging from 0 to 1. Available funds, on the other hand,

[10]Notice, error bars refer to standard deviations.

	X_τ	X_τ	X_τ
w_τ	-0.00013*	-0.00013*	–
	(0.00002)	(0.00002)	
τ	-0.00008	–	-0.00433*
	(0.00135)		(0.0011)
Asset Type	-0.148*	-0.148	-0.149*
	(0.013)	(0.013)	(0.013)
Intercept	0.799*	0.798*	0.802*
	(0.031)	(0.028)	(0.032)
R^2	0.18	0.18	0.15
Adj. R^2	0.17	0.18	0.14
F	63.68	95.62	79.35

Table 3: OLS Regression Results

range from 25 lab dollars to a maximum of 3,463.50 lab dollars (increasing over time, on aggregate). The results of the OLS regression, where share invested in the risky asset is the dependent variable, are reported in Table 3[11]. Recall that available funds, w_τ, and stage, τ, are positively correlated, $\rho_{w_\tau \tau} = 0.601$. The version of the risky asset was transformed into integers, i.e. low, medium, and high risk is equivalent to 1, 2, and 3, respectively. The results of the OLS regression in the second column of Table 3 (best model) corroborate that not only investment shares decrease with rising wealth but also shares are bigger (smaller) for the low (high) risk version.

For these reasons, first, the experimental results are compared to the dynamic stochastic investment predictions based on negative exponential and concave quadratic utility functions, for which optimal investment decisions are functions of current available funds. In addition, the experimental data is

[11] Standard errors are given in parentheses – statistical significance at the one percent level is denoted by *. Number of observations is 891.

evaluated in terms of predictions of constant relative risk aversion. Since optimal investment decisions based on an expo-power utility cannot be obtained algebraically and prospect theory does not explain diversification among cash and risky asset as mentioned in section 3.3.4 and 3.3.5, respectively, only prospect theory is briefly evaluated at the end of this section.

4.1 IRRA utilities

Recall from section 3.3 that both negative exponential and concave quadratic utility functions display IRRA, that is, the share invested in the risky asset decreases with rising wealth.

Table 4 reports the average terminal funds of the 33 period session for all six groups in the experiment in addition to the wealth levels of a negative exponential and concave quadratic expected utility maximizer[12]. In the case of four experimental strings, investing according to the dynamic negative exponential investment strategy yields better results than the average subject in the experiment, especially for rather bad strings. However, two high earnings strings in the experiment did better. Likewise, dynamic investment policy based on concave quadratic utility does better than the average individual in four out of six strings with significant higher earnings for the two lucky strings in the experiment (terminal wealth $> 2,000$ lab dollars). For each dynamic investment policy, only one string yields worse terminal wealth than the 825 lab dollars which participants could be assured of, whereas on average, three groups had less than these 825 lab dollars. Nonetheless, the results indicate that the best investment strategy for the one unlucky streak (first group in session one) would have been to hold all funds in cash and not invest in the risky security.

[12]With parameter values $\eta = 1 \cdot 10^{-3}$ and $b = 2.5 \cdot 10^{-4}$ $(5 \cdot 10^{-4})$ for experimental strings with $\max(w) < 4,000\,(2,000)$ lab dollars.

	Session 1		Session 2		Session 3	
Experiment	223.18 (132.40)	2,146.12 (1,181.01)	817.17 (744.63)	1,330.77 (870.39)	580.28 (309.51)	2,011.52 (416.23)
Neg. Expo.	647.15	1,798.50	1,039.79	1,423.42	863.85	1,784.85
Quadratic	180.90	3,019.09	1,130.89	1,011.49	887.09	2,671.92

Table 4: Average Terminal Wealth in Lab Dollars

Figures 9 through 14 in Appendix B plot the predicted investment shares over time along with the average fractions from experiment. Overall, predicted dynamic investment shares are bigger at earlier stages compared with experiment. At later stages of the life cycle experiment, however, participants invested larger fractions of their funds in the medium and high version of the risky security than negative exponential and concave quadratic utility suggest. Moreover, toward the end of the 33 period session, average subjects invested smaller shares in the low risk version of the security, whereas both dynamic investment policies suggest to invest all funds in the low risk security.

Although stochastic dynamic optimization policies from above display IRRA as observed in the experiment, the plots indicate that aggregate observed behavior is not consistent with predictions. However, notice importantly, that the groups were rather small and therefore standard deviations (denoted by error bars) are big. In Figure 15 (see Appendix B, the aggregate predictions (all six strings) and average shares from experiment are plotted with errors bars denoting standard errors. Z-statistics test reveals that neither the differences between negative exponential dynamic optimization and experiment nor between concave quadratic utility and experiment are statistically significant at the five percent level. However, the samples are simply too small in order to make a definite statement about the applicability of the proposed negative exponential and concave quadratic utility as descriptive risk averse utility function.

4.2 CRRA

CRRA implies that the share invested in the risky security is constant for each version; however, the absolute amount invested increases with rising wealth. Table 5 lists the optimal fractions X_R^*, which an investor would put into the low, medium, or high risk security based on expected utility maximization with the power utility function as descriptive risk averse utility function (41) for different parameter values of $\alpha = \tilde{r}$. Notice that a risk neutral investor ($\alpha = 0$) would plunge all available funds in the risky security since the expectation value of the lottery exceeds the sure thing (cash holding).

	$\alpha = 0.7$	$\alpha = 0.6$	$\alpha = 0.5$	$\alpha = 0.4$
Low	1	1	1	1
Medium	0.24	0.28	0.33	0.42
High	0.08	0.09	0.11	0.14

Table 5: Optimal Investment Shares of a CRRA Power Utility

Aggregate behavior of all 27 participating subjects is compared with the predictions of CRRA since $X_\tau = X$ is independent of current wealth, and should therefore not be influenced by a certain streak of lucky or unlucky draws. Figure 16 in Appendix B graphically compares the investment shares from Table 5 with aggregate behavior observed in the experiment. The Figure shows that on aggregate, investment decisions in the experiment are inconsistent with any of the CRRA levels. Interestingly, the expected utility maximization of a CRRA utility suggests investment of all funds in the low risk version of the risky security at all levels of CRRA, whereas aggregate experimental shares are well below. On the other hand, CRRA underpredicts the shares to be invested in the medium or high risk security when compared with experiment.

4.3 Prospect Theory

Recall from section 3.3.5 that the investment decision of a value maximizer depends on the relationship between good and bad return, decision weights, and parameters κ and τ, i.e. the good return is an increasing function of the bad return, ceteris paribus. In Table 6, hypothetical good yields of the risky security are shown which satisfy that an investor would be indifferent between cash holdings and investment in the risky security for different parameter values of κ, given R_b and $\tau = 0.9$. Based on the assumption that an investor dislikes a loss roughly twice as much as she likes a gain [Tve92], she would never invest in the risky security, given the combination of yields in the experiment. However, none of the individuals participating in our laboratory experiment did so, which indicates that the previous assumption of participating subjects being expected utility rather than value maximizers is true.

κ	2.25	2.00	1.75	1.5	1.25
$R_b = -0.1$	0.24	0.22	0.19	0.16	0.13
$R_b = -0.5$	1.23	1.08	0.93	0.78	0.64
$R_b = -0.9$	2.22	1.94	1.68	1.41	1.15

Table 6: R_g as a function of R_b, for different values of κ, $\tau = 0.9$

5 Conclusion

In this paper, I compare a laboratory experiment with stochastic dynamic optimization policies based on expected utility theory. Specifically, I focus on the predictions based on a negative exponential and concave quadratic utility. In addition, I look at the applicability of a rather exotic functional form [Sah93] to the experimental two security portfolio problem. Prospect theory is reviewed critically as descriptive risk averse decision model. In their work, Kahneman and Tversky [Kah79] present several choice problems where preferences violate the axioms of expected utility theory. They draw the conclusion that expected utility theory is not a descriptive model of decision making when facing risk. Being aware of the limitations of expected utility theory and respective absurd predictions due to concave utility functions [Rab00], based on the experimental findings, I can neither support nor reject the hypothesis by Rabin et al. [Rab01], which states that expected utility theory is not the right explanation for most risk attitudes.

However, the findings imply that CRRA is inconsistent with risk aversion in terms of a life cycle investment problem. Additionally, prospect theory does not explain why risk averse investors allocate some of their funds in the risky security and concomitantly hold parts in cash (exclusion of diversification as investment principle).

Notice importantly, that in order to make definite statements, a larger experimental sample is necessary. Furthermore, the experimental design should be reviewed, i.e. in order for the law of large numbers to apply, it is better to determine the random outcome of the risky security individually rather than for the whole group. This can be done by extending the experiment by a random number generator. Although this can be easily implemented, one has to be cautious since participating subject may or may not perceive a random number generator more critically than the physical toss of a coin.

A Maple Source Codes

In the following, the Maple® 8 dynamic programming source codes are given for the negative exponential (concave quadratic) utility. Useful techniques to solve dynamic stochastic optimization problems can be found for instance in [Car06] and [Par00].

A.1 Dyn-Expo.mws

```
> restart: # Dyn-Quadratic.mws
> # declare vector of optimal decisions and probabilities
> mu:=array(1..33);
> p:='p':q:='q';
> # value function at end of life cycle
> M[0]:=w->-exp(-n*w)/n;
> E[1]:=-(p*exp(-n*(w+w*X*R[32]))
+q*exp(-n*(w-w*X*Q[32])))/(r*n);
> # find optimal share to invest mu[1] at end
> d[1]:=normal(diff(E[1],X));
> mu[1]:=solve(d[1],X);
> # express value function in terms of mu[1]
> M[1]:=unapply(expand(simplify(subs(X=mu[1],E[1]))),w);
> # recursive calculation for all other stages
> for i from 2 to 33 do:
> E[i]:=(p*M[i-1](w+w*X*R[33-i])+q*M[i-1](w-w*X*Q[33-i]))/r:
> d[i]:=normal(diff(E[i],X)):
> mu[i]:=normal(solve(d[i],X)):
> M[i]:=unapply(expand(normal(subs(X=mu[i],c[i]))),w):
> end do: print(mu);
```

A.2 Dyn-Quadratic.mws

```
> restart: # Dyn-Quadratic.mws
> mu:=array(1..33): p:='p':q:='q';
> M[0]:=w->w-b*w^2/2;
> E[1]:=(p*(w+w*X*R[32]-b/2*(w+w*X*R[32])^2)
+q*(w-w*X*Q[32]-b/2*(w-w*X*Q[32])^2))/r;
> d[1]:=normal(diff(E[1],X));
> mu[1]:=solve(d[1],X);
> M[1]:=unapply(expand(simplify(subs(X=mu[1],E[1]))),w);
> for i from 2 to 33 do:
> E[i]:=(p*M[i-1](w+w*X*R[33-i])+q*M[i-1](w-w*X*Q[33-i]))/r:
> d[i]:=normal(diff(E[i],X)):
> mu[i]:=normal(solve(d[i],X)):
> M[i]:=unapply(expand(normal(subs(X=mu[i],E[i]))),w):
> end do: print(mu);
```

B Graphs and Tables

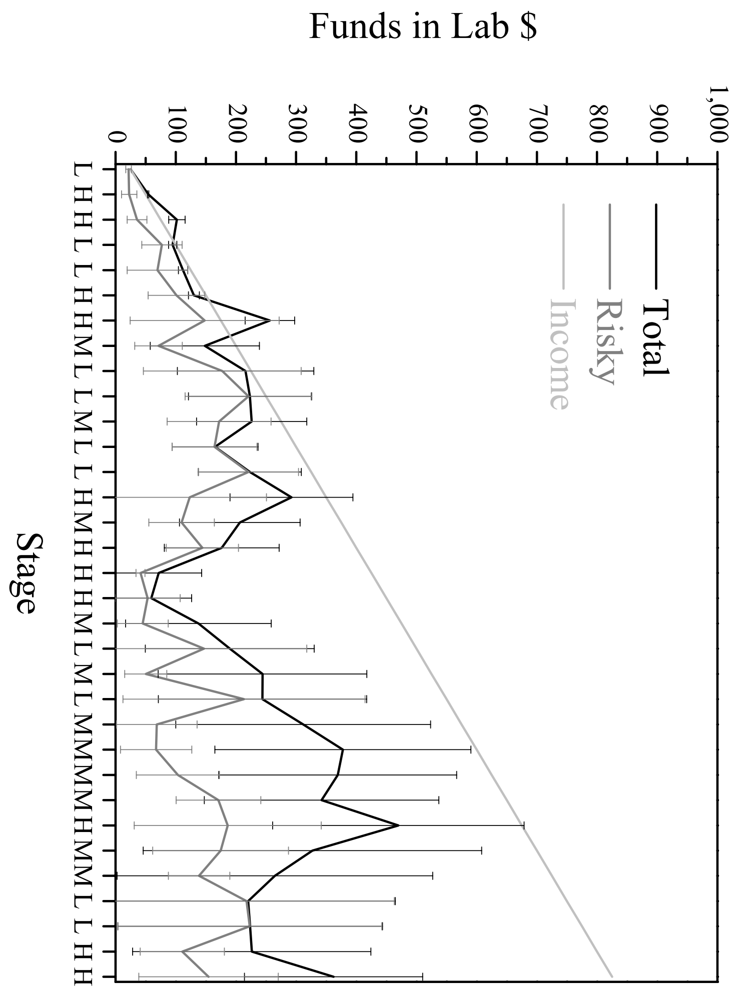


Figure 3: Total and Invested Funds (First Session, First Group)

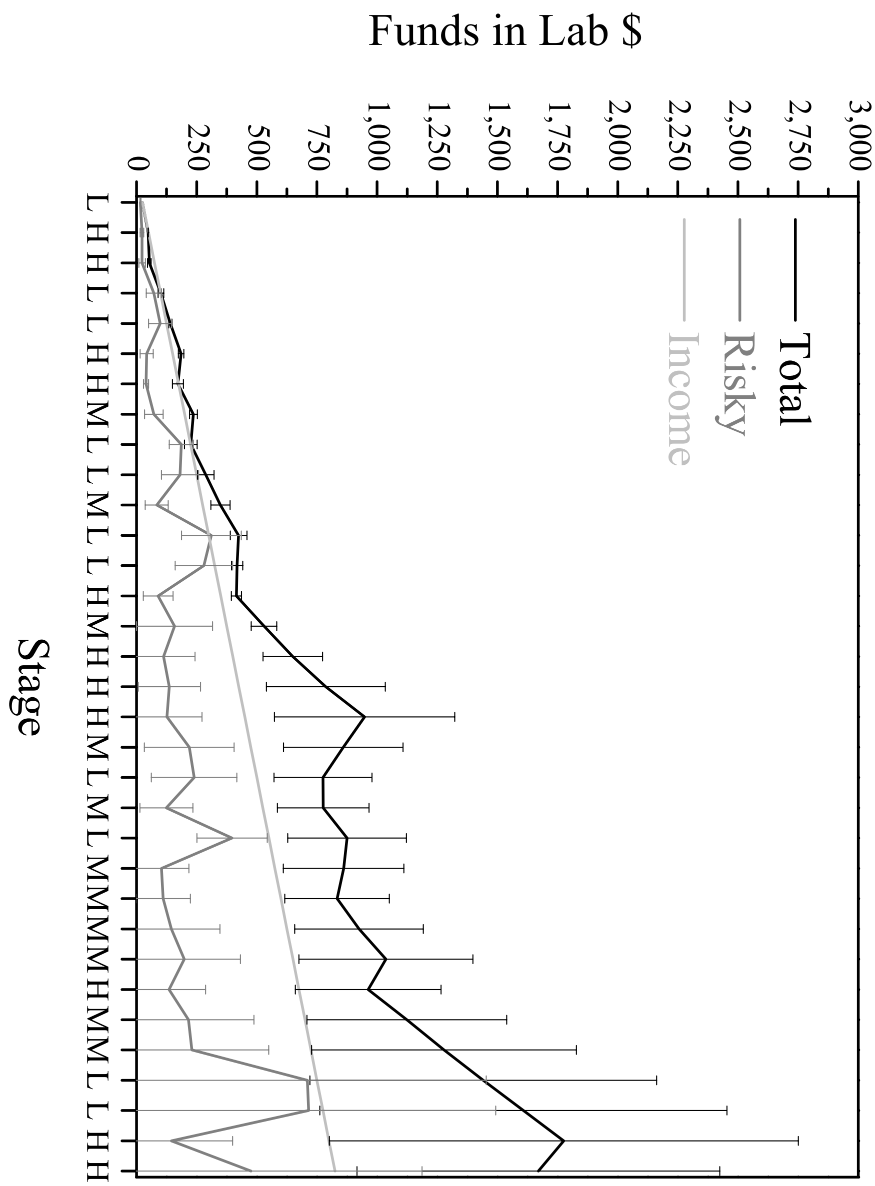


Figure 4: Total and Invested Funds (First Session, Second Group)

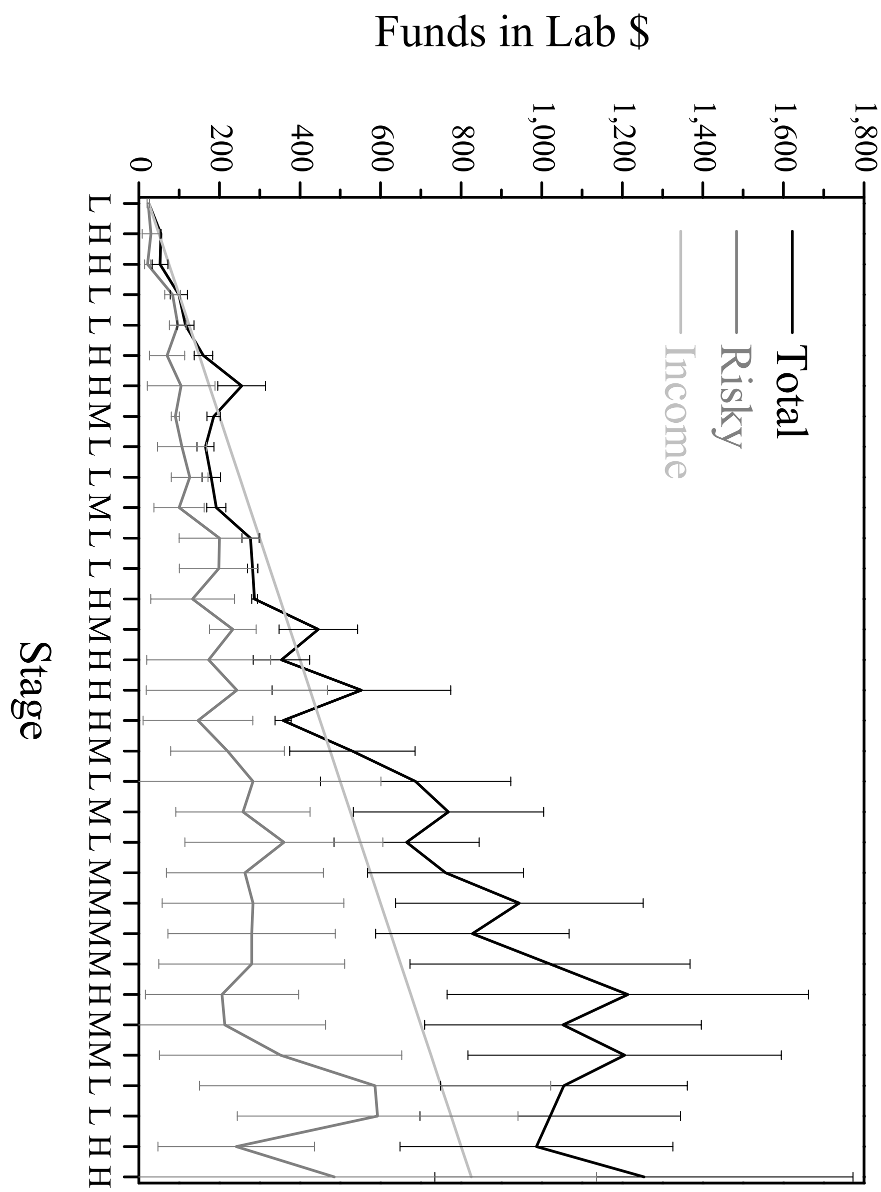


Figure 5: Total and Invested Funds (Second Session, First Group)

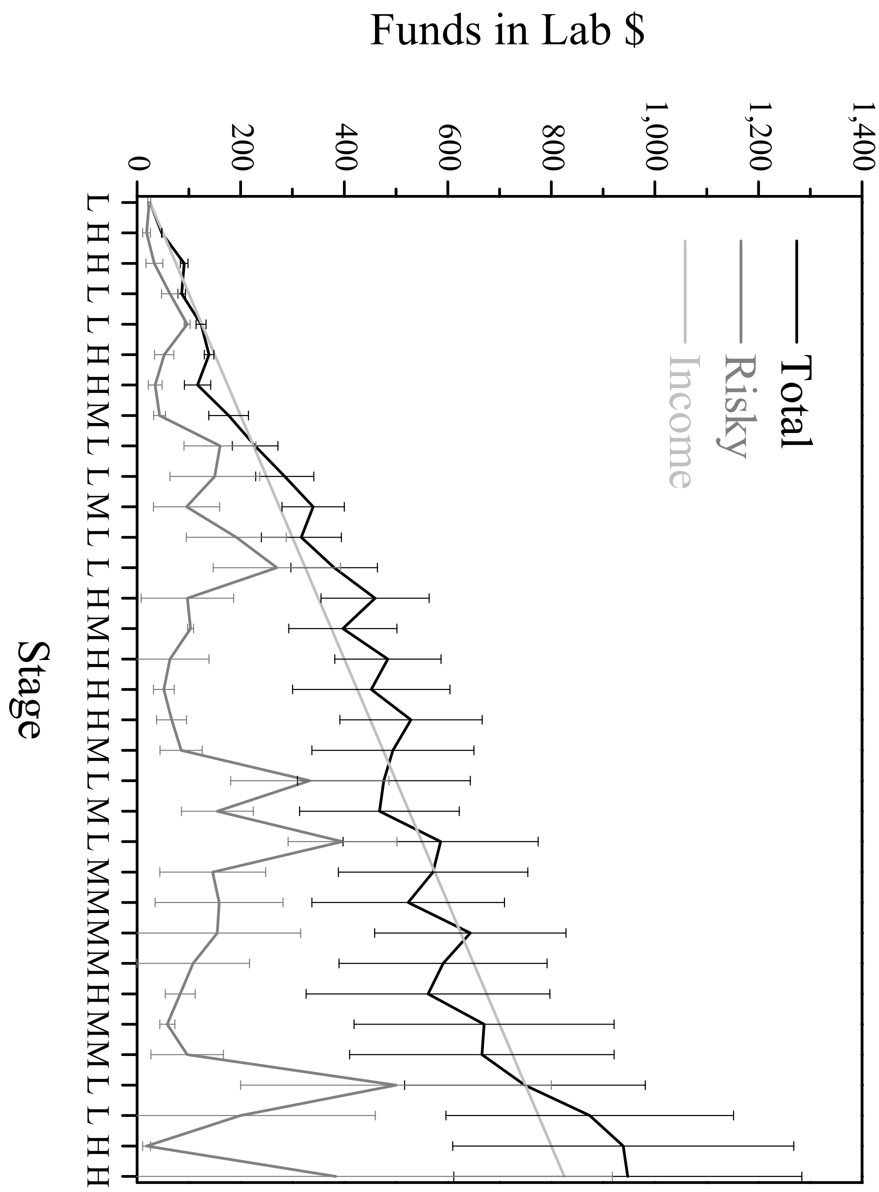


Figure 6: Total and Invested Funds (Second Session, Second Group)

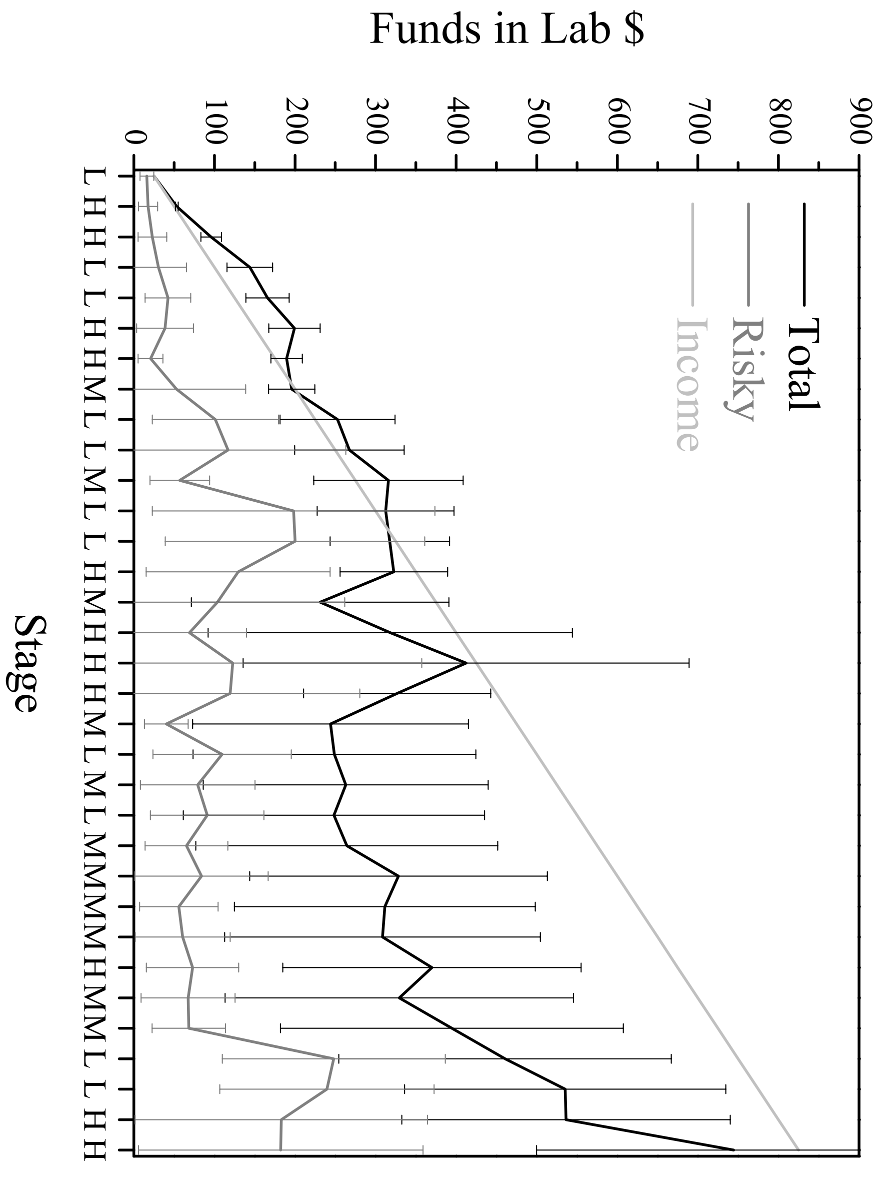


Figure 7: Total and Invested Funds (Third Session, First Group)

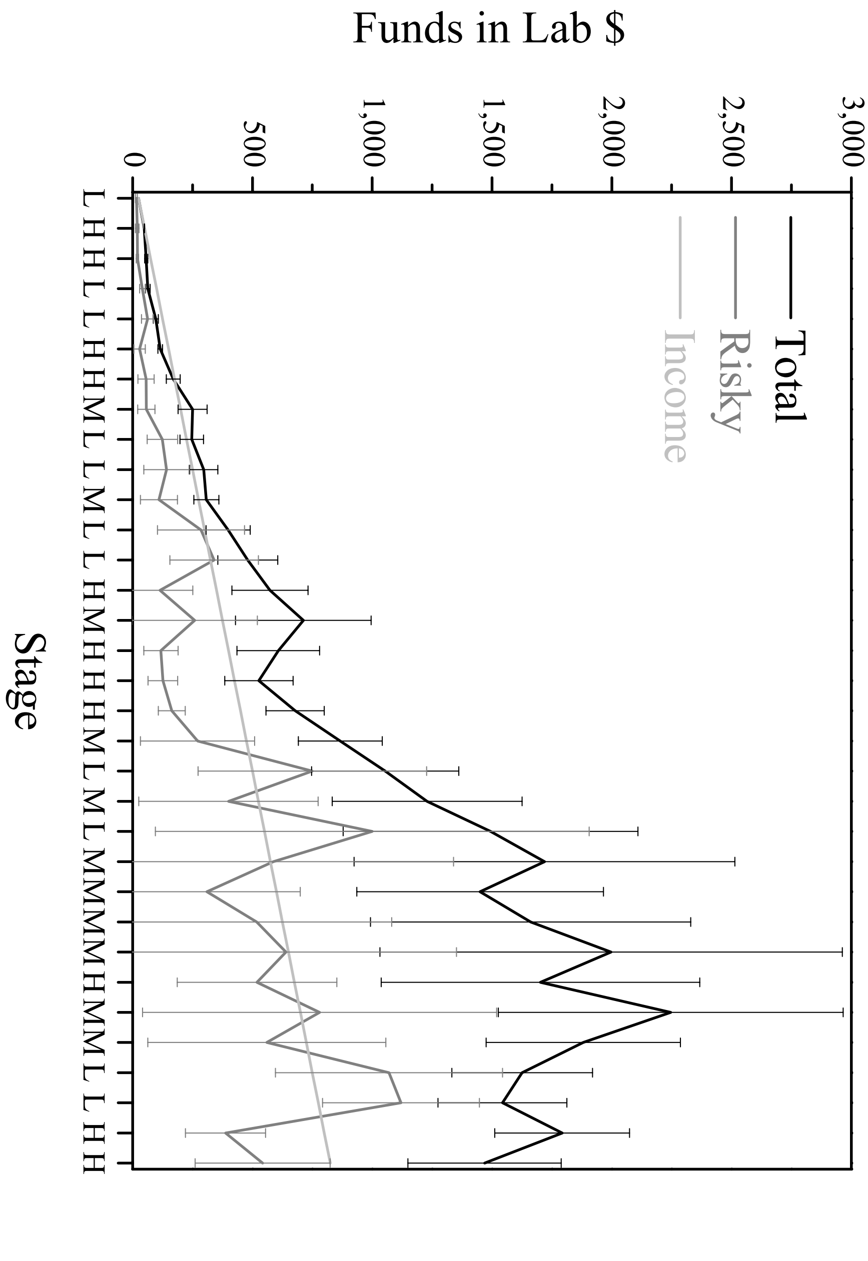


Figure 8: Total and Invested Funds (Third Session, Second Group)

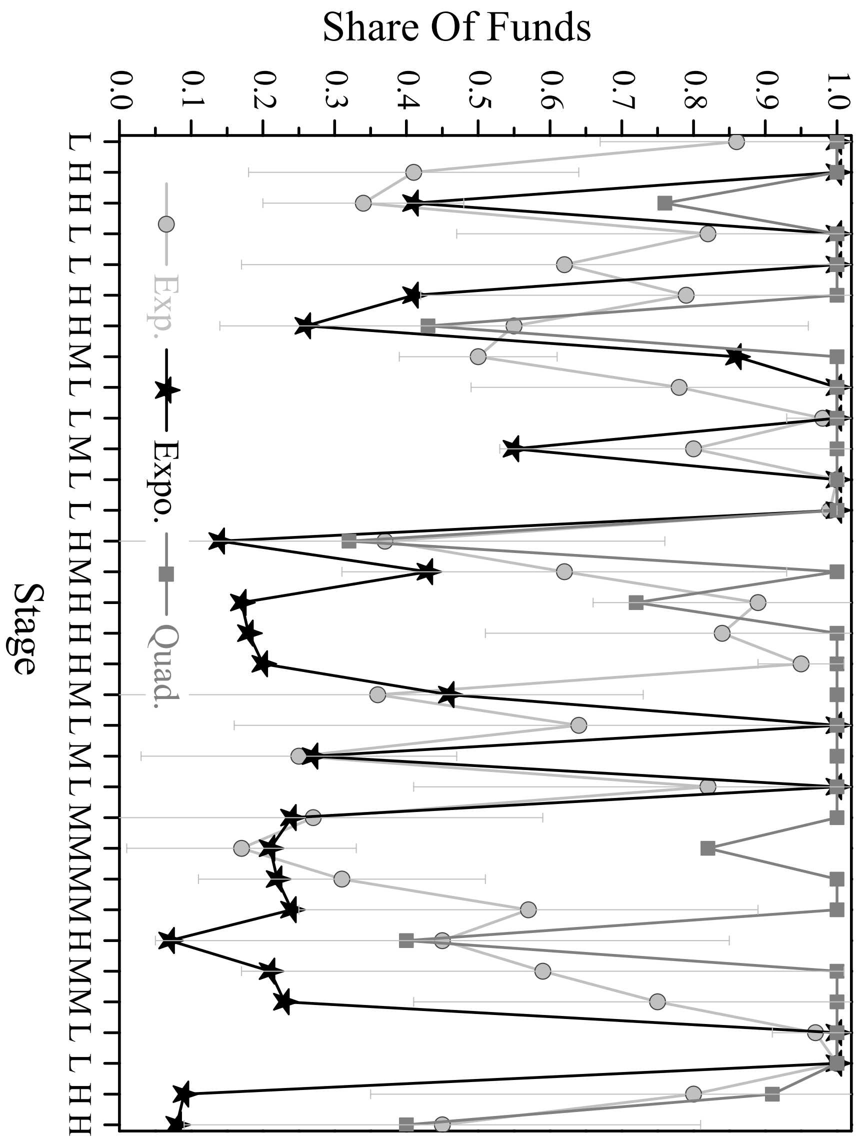


Figure 9: Investment Shares (First Session, First Group)

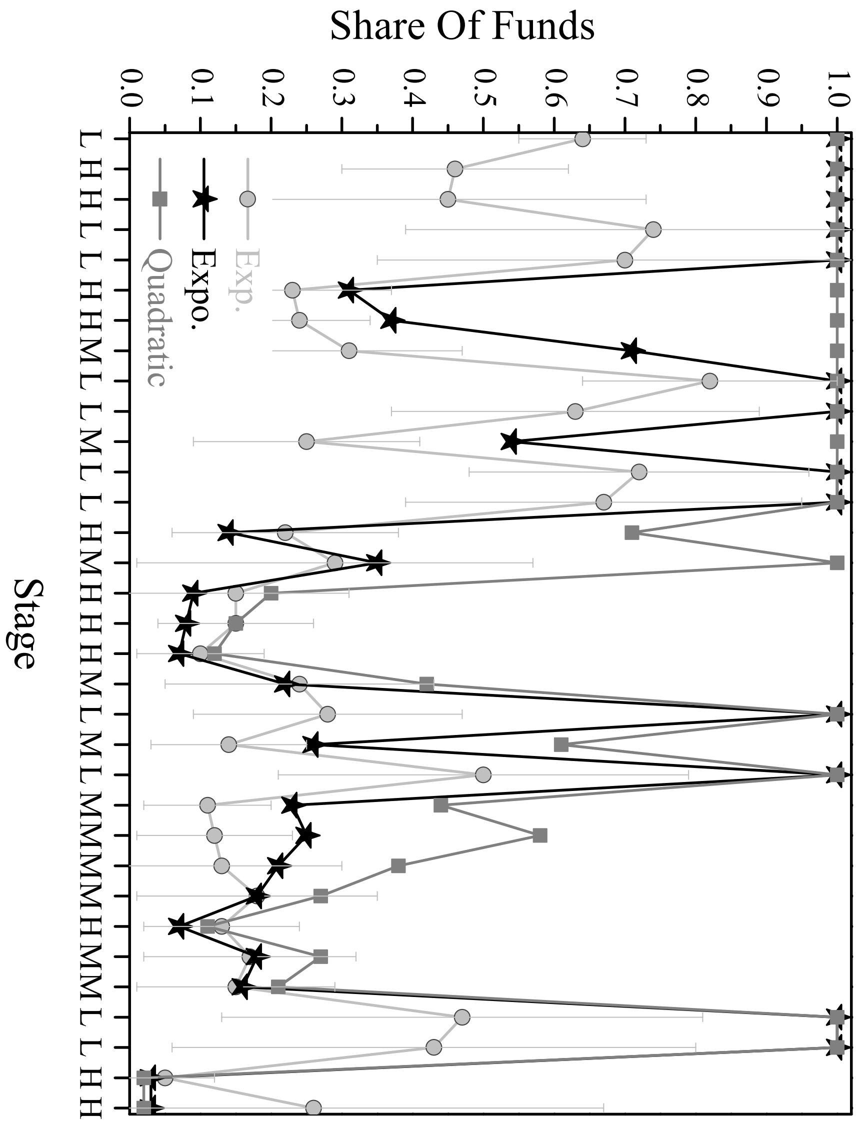


Figure 10: Investment Shares (First Session, Second Group)

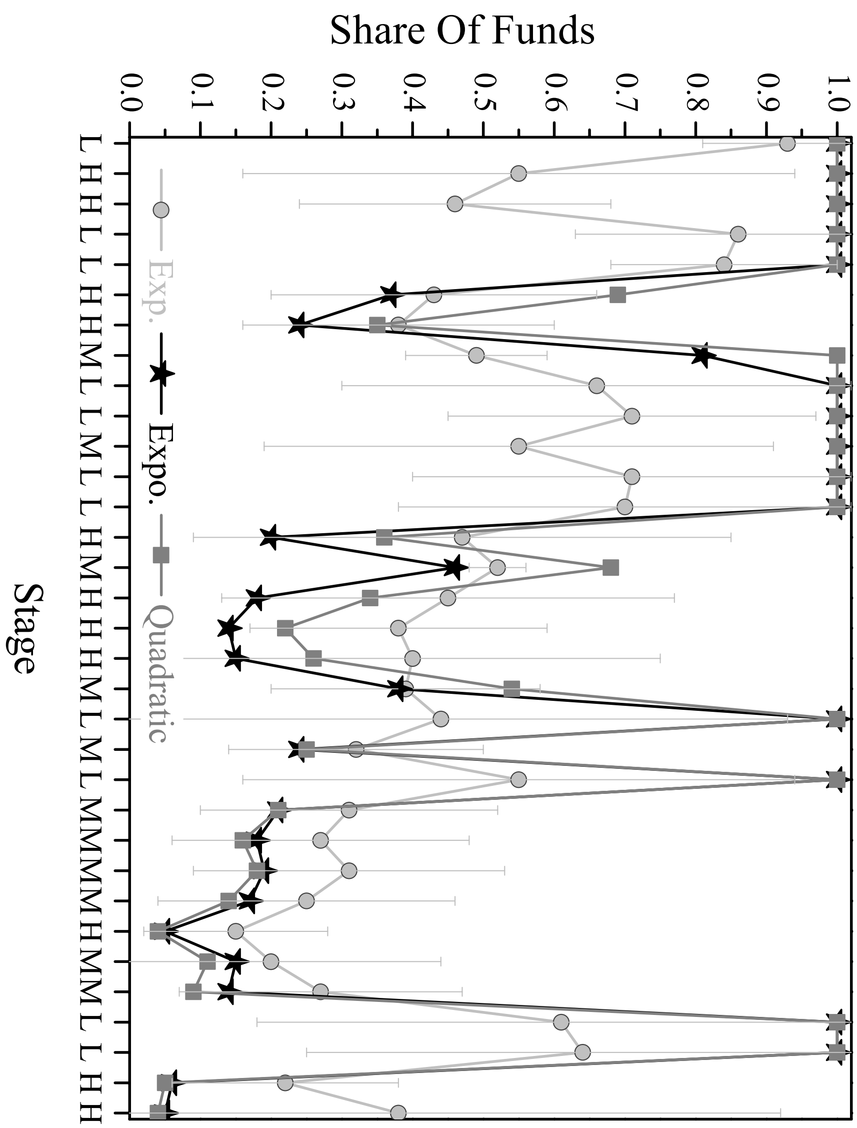


Figure 11: Investment Shares (Second Session, First Group)

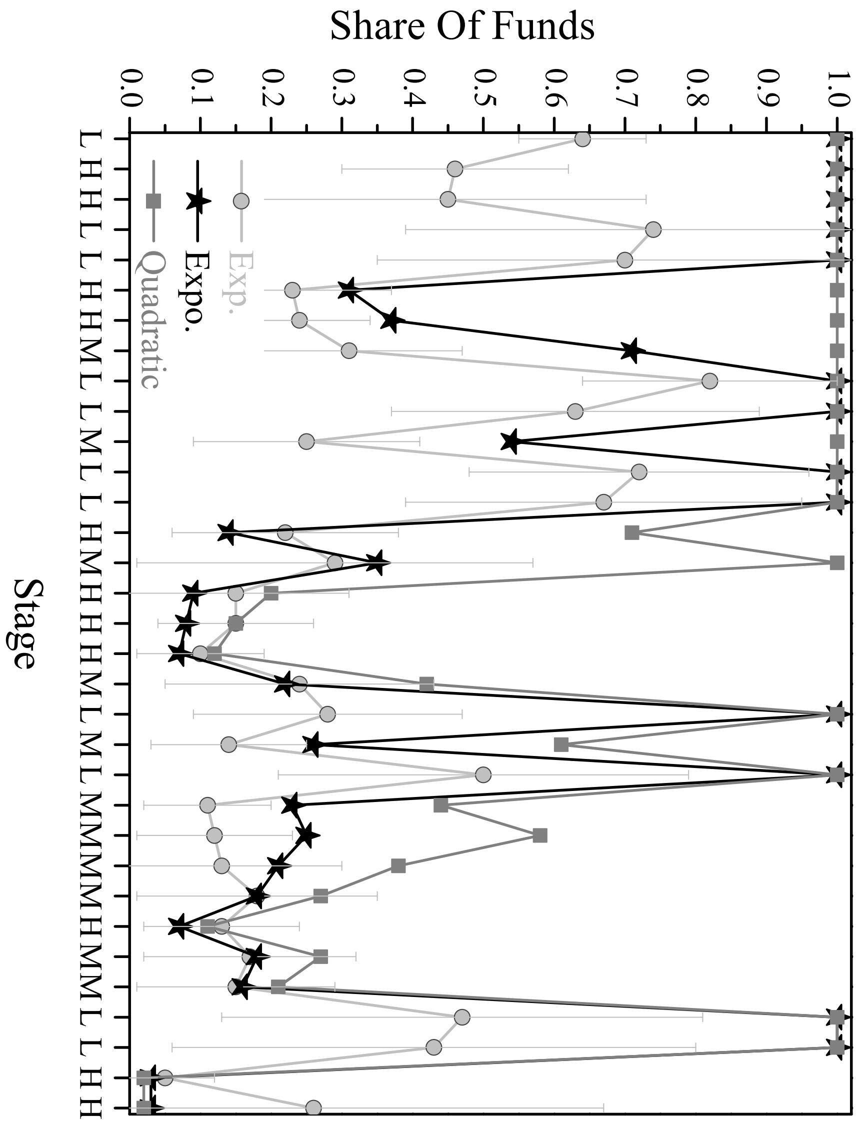


Figure 12: Investment Shares (Second Session, Second Group)

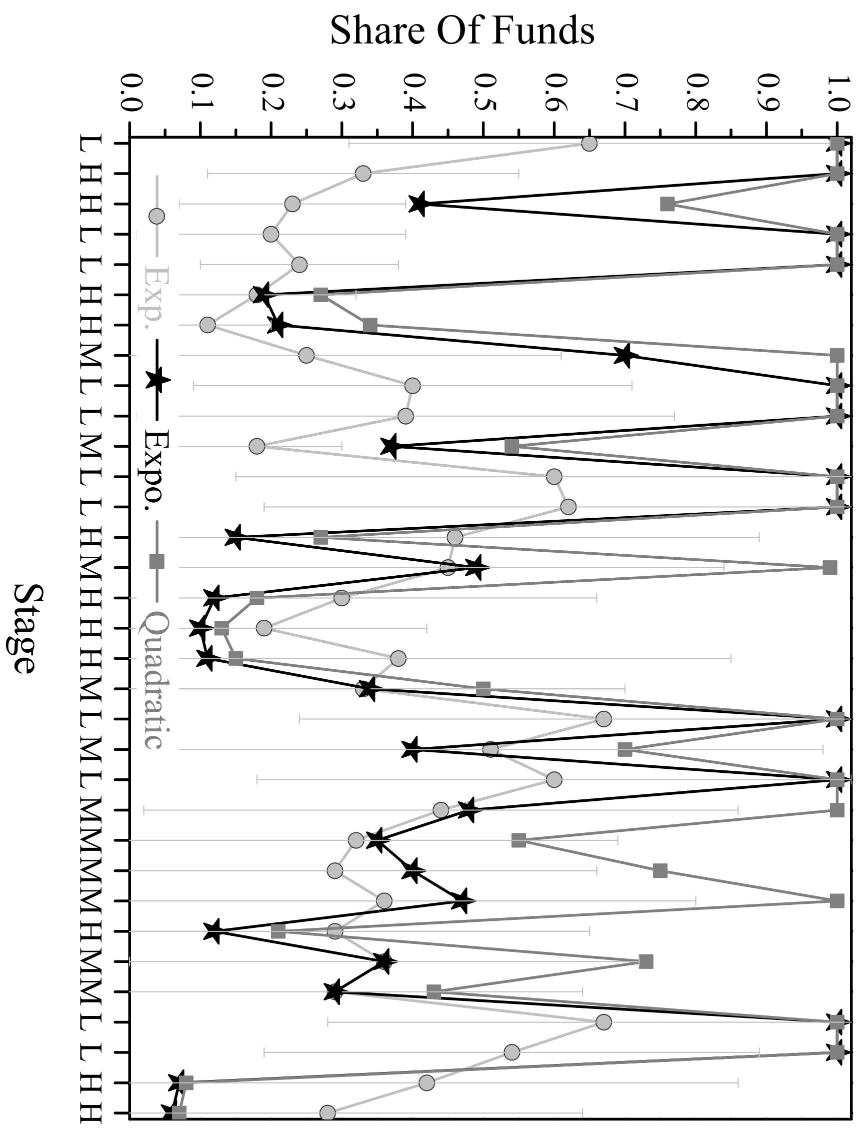


Figure 13: Investment Shares (Third Session, First Group)

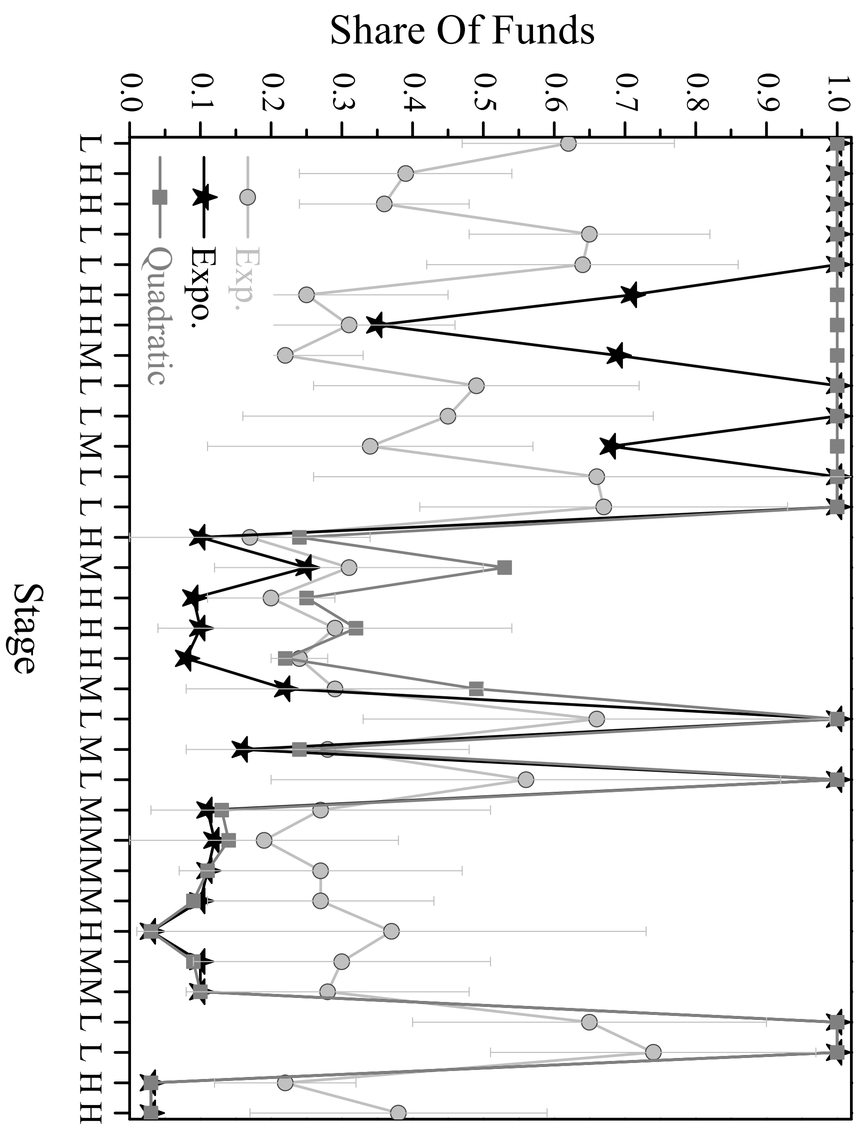


Figure 14: Investment Shares (Third Session, Second Group)

		First Session		Second Session		Third Session	
Stage	Type	$n = 5$	$n = 5$	$n = 3$	$n = 3$	$n = 6$	$n = 5$
0	L	Good	Bad	Good	Bad	Good	Bad
1	H	Good	Bad	Bad	Good	Good	Bad
2	H	Bad	Good	Good	Bad	Good	Bad
3	L	Bad	Good	Bad	Good	Bad	Good
4	L	Bad	Good	Good	Bad	Good	Bad
5	H	Good	Bad	Good	Bad	Bad	Good
6	H	Bad	Good	Bad	Good	Bad	Good
7	M	Good	Bad	Bad	Good	Good	Bad
8	L	Bad	Good	Bad	Good	Bad	Good
9	L	Bad	Good	Bad	Good	Good	Bad
10	M	Bad	Good	Good	Bad	Bad	Good
11	L	Good	Bad	Bad	Good	Bad	Good
12	L	Good	Bad	Bad	Good	Bad	Good
13	H	Bad	Good	Good	Bad	Bad	Good
14	M	Bad	Good	Bad	Good	Good	Bad
15	H	Bad	Good	Good	Bad	Good	Bad
16	H	Bad	Good	Bad	Good	Bad	Good
17	H	Good	Bad	Good	Bad	Bad	Good
18	M	Good	Bad	Good	Bad	Bad	Good
19	L	Good	Bad	Good	Bad	Bad	Good
20	M	Bad	Good	Bad	Good	Bad	Good
21	L	Good	Bad	Good	Bad	Bad	Good
22	M	Good	Bad	Good	Bad	Good	Bad
23	M	Bad	Good	Bad	Good	Bad	Good
24	M	Bad	Good	Good	Bad	Bad	Good
25	M	Good	Bad	Good	Bad	Good	Bad
26	H	Good	Bad	Bad	Good	Bad	Good
27	M	Bad	Good	Good	Bad	Good	Bad
28	M	Bad	Good	Bad	Good	Good	Bad
29	L	Bad	Good	Bad	Good	Good	Bad
30	L	Bad	Good	Bad	Good	Bad	Good
31	H	Good	Bad	Good	Bad	Good	Bad
32	H	Bad	Good	Bad	Good	Bad	Good

Table 7: Experimental Strings

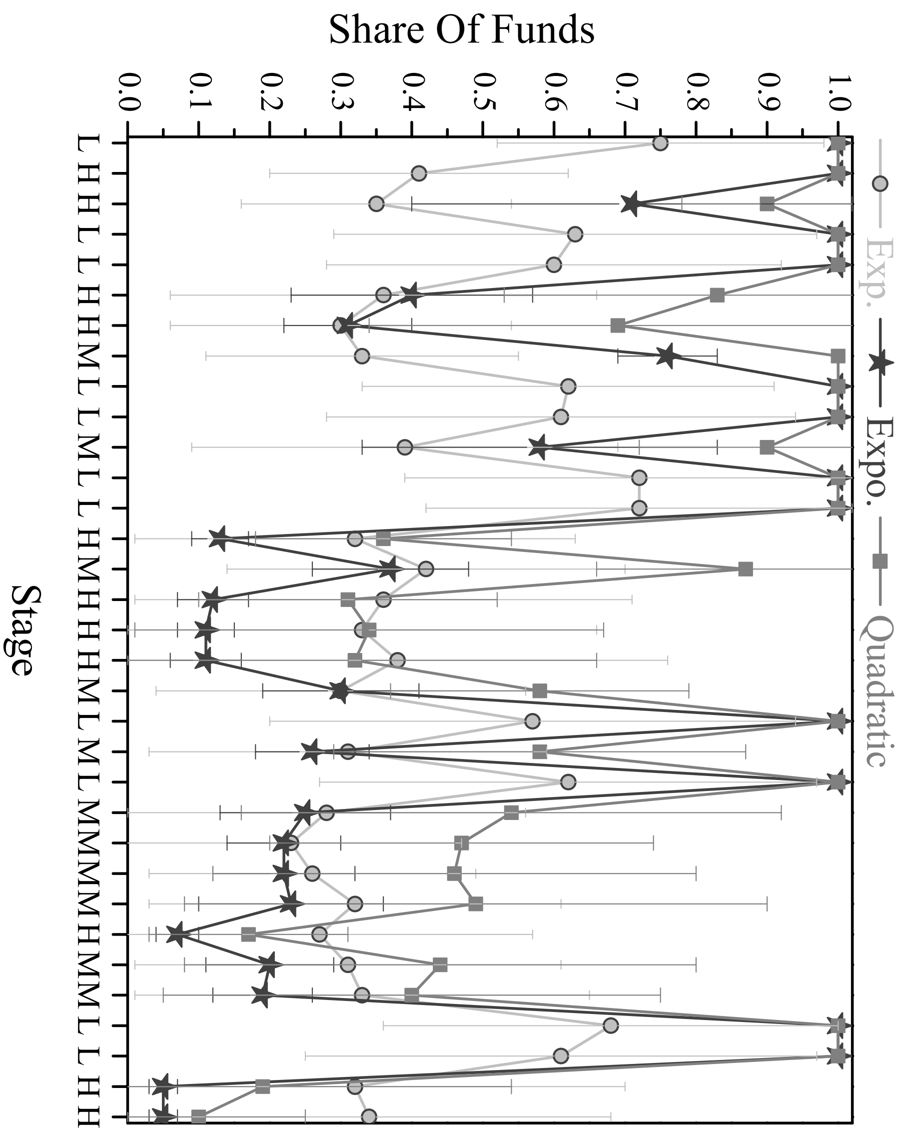


Figure 15: Aggregate Shares, All Sessions

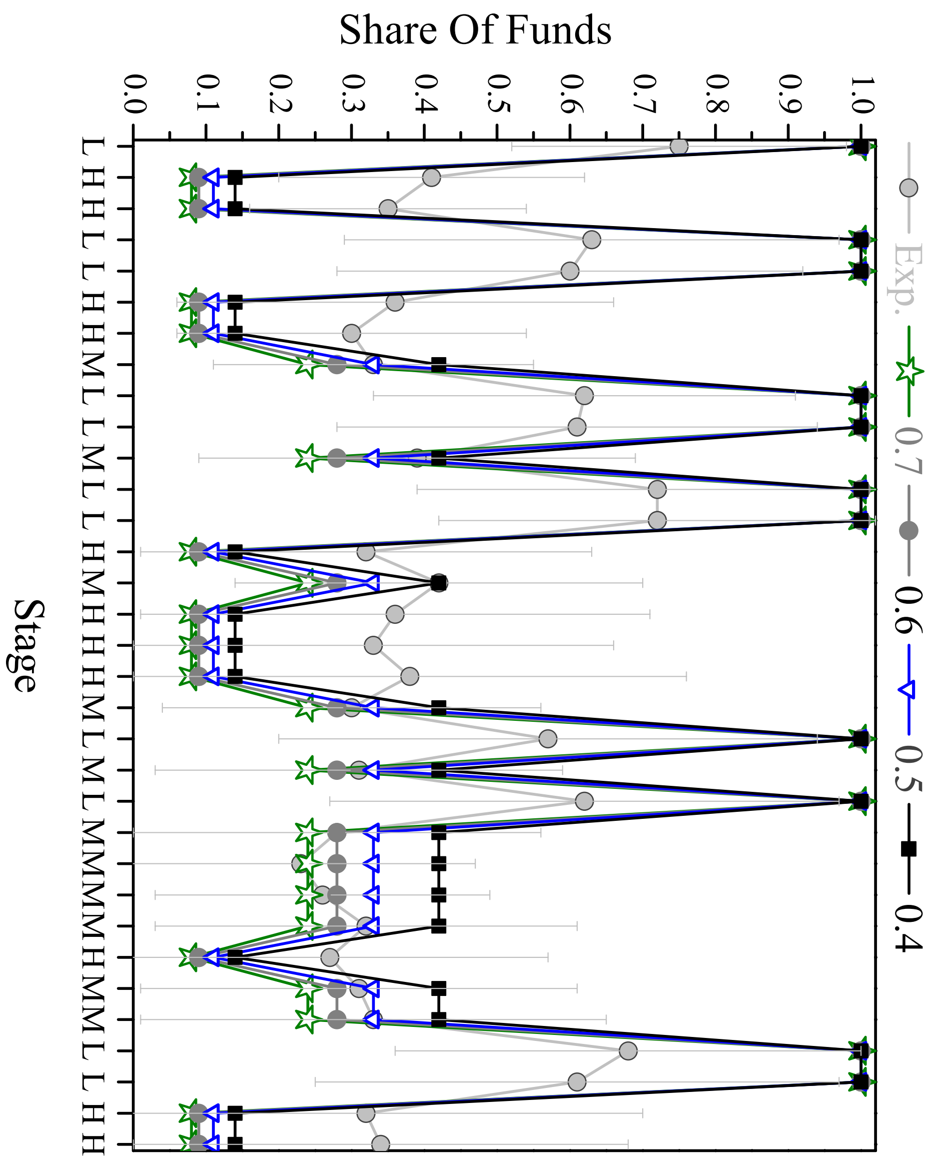


Figure 16: CRRA Shares

References

[Car06] Carroll, Christopher, D.. "Lecture Notes On Solution Methods for Microeconomic Dynamic Stochastic Optimization Problems" 2006. John Hopkins University, Department of Economics. 20 Jan. 2006 <http://econ.jhu.edu/people/ccarroll/SolvingMicroDSOPs.pdf>.

[Ewi04] Ewing, Bradley and Brown Kruse, Jamie. "Asset Allocation and Risk: Laboratory Experiments" Working Paper, Texas Tech U, Dep. of Economics, 2004.

[Hol02] Holt, Charles A., Laury, Susan K.. "Risk Aversion and Incentive Effects"The American Economic Review 92, 5 (2002): 1644-1655.

[Hua88] Huang Chi-fu, Litzenberger, Robert H.. *Foundations For Financial Economics.* New York: North-Holland, 1988.

[Ing87] Ingersoll, Jonathan E. Jr.. *Theory of Financial Decision Making.* New Jersey: Rowman & Littlefield, 1987.

[Jeh01] Jehle, Geoffrey, A., Reny, Philip, R.. *Advanced Microeconomic Theory.* Boston: Addison Wesley, 2001.

[Kah79] Kahneman, Daniel and Tversky, Amos. "Prospect Theory: An Analysis of Decision Under Risk" Econometrica 47, 2 (1979): 263-291.

[Mar91] Markowitz, Harry M.. *Portfolio Selection - Efficient Diversification of Investments.* Cambridge: Basil Blackwell, 1991.

[Mar52] Markowitz, Harry M.. "Portfolio Selection" The Journal of Finance 7 (1952): 77-91.
– "The Utility of Wealth" The Journal of Political Economy 60, 2 (1952): 151-158.

[Mer69] Merton, Robert C.. "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case" The Review of Economics and Statistics 51, 3 (1969): 247-257.

[Par00] Parlar, Mahmut. *Interactive Operations Research with Maple: Methods and Models.* Boston: Birkhäuser, 2000.

[Pra64] Pratt, John W.. "Risk Aversion in the Small and in the Large." Econometrica 32, 1-2 (1964): 122-136.

[Rab01] Rabin, Matthew and Thaler, Richard H.. "Anomalies - Risk Aversion" Journal of Economic Perspectives 15, 1 (2001): 219-232.

[Rab00] Rabin, Matthew. "Risk Aversion and Expected-Utility Theory: A Calibration Theorem" Econometrica 81, 5 (2000): 1281-1292.

[Sah93] Saha, Atanu. "Expo-Power Utitlity: A "Flexible" Form for Absolute and Relative Risk Aversion" American Journal of Agricultural Economics 75, 4 (1993): 905-913.

[Sam69] Samuelson, Paul A.. "Lifetime Portfolio Selection By Dynamic Stochastic Programming" The Review of Economics and Statistics 51, 3 (1969): 239-246.

[Sam63] Samuelson, Paul A.. "Risk and Uncertainty: A Fallacy of Large Numbers." Scientia 98 (1963): 108-113.

[Tob58] Tobin, J.. "Liquity Preference as Behavior Towards Risk" The Review of Economic Studies XXV (1958): 65-86.

[Tve92] Tversky, Amos, Kahneman, Daniel. "Advances in Prospect Theory Cumulative Representation of Uncertainty" Journal of Risk and Uncertainty, V (1992): 297-323.